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Synchronous behavior in asymmetrically coupled pendulums

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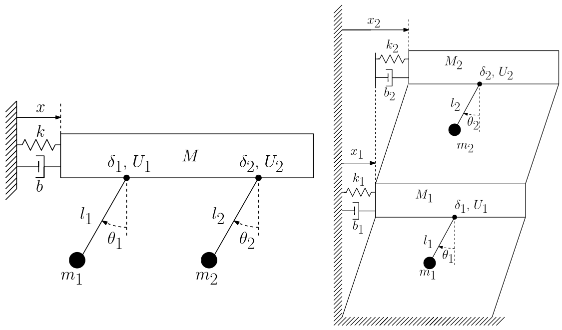
Abstract—It is well-known that a pair of pendulum-like oscillators, placed on a suspended rigid bar, may exhibit in-phase or anti-phase synchronized motion. Here, a novel coupling structure, in which the pendulums are asymmetrically coupled, is presented. Due to the physics underlying the dynamics of the coupling, the pendulums do not achieve complete in-phase or anti-phase synchronization. Instead, the pendulums oscillate at the same frequency but with different amplitudes and with a phase difference close to π or zero. The amplitude, phase, and frequency of the synchronous solutions, are determined by using the Poincaré method of perturbation and the obtained results are illustrated by means of numerical simulations.

1. Introduction

Around 1665, the Dutch scientist Christiaan Huygens made an exciting discovery while working on finding a solution for the longitude problem [1]. Huygens observed that two of his recently invented pendulum clocks, which were hanging from a wooden beam placed on the top of two chairs, were showing and 'odd kind of sympathy': the pendulums of the clocks were oscillating in synchrony. After a systematic experimental study, Huygens did realize that the key element in his experiment was the wooden bar, i.e. the coupling structure[2]. Nowadays, the phenomenon discovery by Huygens is called Huygens' synchronization, and the wooden bar in the top of two chairs is referred to as Huygens' coupling, cf. [3, 4].

One of the simplest models used in the study of Huygens' synchronization is depicted in Figure 1a. In this model, the clocks have been replaced by self-sustained pendulums and the wooden beam is modelled as a rigid bar elastically attached to a fixed support. Note that, in this case, the coupling between the pendulums is symmetric.

In this paper, a novel coupling scheme—closely related to Huygens' experiment—is presented. In contrast to the configuration presented in Figure 1a, the coupling structure introduced here consists of a two-level building-like structure, as depicted in Figure 1b, in which the pendulums are asymmetrically coupled. The onset of synchronous solutions in the system is investigated by using the Poincaré method, which allows to determine the amplitude, frequency, and phase of the synchronous solutions and also, it can be used for investigating the stability of the solutions.



(a) (b)

Figure 1: a) Symmetric Huygens' coupling [5]. b) Proposed asymmetric coupling.

The outline of the paper is as follows. The model for the proposed system of coupled pendulums is presented in Section 2. Next, in Section 3, the occurrence of synchronization is investigated by using the Poincaré method. Then, a numerical study is conducted in Section 4. Finally, Section 5 presents some conclusions.

2. Model

Consider the schematic model depicted in Figure 1b. Each level of the building-like structure is modelled by a mass-spring-damper-system, with mass M_j [kg], stiffness coefficient k_j [N/m] and damping coefficient b_j [Ns/m], for $j = 1, 2$. On the other hand, each pendulum is modelled as a point mass of mass m_j [kg], attached to the end of a massless rod of length l_j [m], for $j = 1, 2$. Furthermore, it is assumed that the rotational damping in pendulum j is viscous, with damping coefficient δ_j [Nms/rad], and moreover, each pendulum is driven by the torque U_j, for $j = 1, 2$.

The mathematical model, obtained using the Lagrangian formalism, is given by

$$\begin{aligned}
\ddot{x}_1 l_1^{-1}\cos(\theta_1) + \ddot{\theta}_1 &= \eta_1 U_1 - \omega_1^2 \sin(\theta_1) - \xi_1\dot{\theta}_1,\\
\ddot{x}_2 l_2^{-1}\cos(\theta_2) + \ddot{\theta}_2 &= \eta_2 U_2 - \omega_2^2 \sin(\theta_2) - \xi_2\dot{\theta}_2,\\
\ddot{x}_1 + \beta_1\ddot{\theta}_1\cos(\theta_1) &= \beta_1\dot{\theta}_1^2\sin(\theta_1) + \varphi_1/M_{T1},\\
\ddot{x}_2 + \beta_2\ddot{\theta}_2\cos(\theta_2) &= \beta_2\dot{\theta}_2^2\sin(\theta_2) + \varphi_2/M_{T2},
\end{aligned} \tag{1}$$

where $x_j \in \mathbb{R}$ is the horizontal displacement of platform j and $\theta_j \in \mathbb{R}$ is the angular displacement of pendulum j,

$\omega_j = \sqrt{g/l_j}$ [rad/s], $g = 9.81$ [m/s^2], $\xi_j = \delta_j(m_j l_j^2)^{-1}$, $\eta_j = (m_j l_j^2)^{-1}$, $\beta_j = m_j l_j M_{Tj}^{-1}$, $M_{Tj} = M_j + m_j$, for $j = 1, 2$, $\varphi_1 = -k_1 x_1 + k_2(x_2 - x_1) - b_1\dot{x}_1 + b_2(\dot{x}_2 - \dot{x}_1)$ and $\varphi_2 = -k_2(x_2 - x_1) - b_2(\dot{x}_2 - \dot{x}_1)$. In this study, it is assumed that the driving torque is modelled by the van der Pol term $U_j = \nu_j(\gamma_j^2 - \theta_j^2)\dot{\theta}_j$ where $\nu_j \in \mathbb{R}_+$ is the amount of nonlinearity and $\gamma_j \in \mathbb{R}_+$ defines the switching between positive and negative damping, for $j = 1, 2$.

3. Theoretical analysis

The existence of synchronous solutions in system (1) is investigated by using the Poincaré method [6, 7]. For the analysis, the following assumptions are considered: **[A1]**: the pendulums and the platforms are identical, i.e. $m_j = m$, $l_j = l$, $\delta_j = \delta$, $\nu_j = \nu$, $\gamma_j = \gamma$, $M_j = M$, $k_j = k$, and $b_j = b$, for $j = 1, 2$. **[A2]**: the angular displacements of the pendulums is assumed to be small, such that $\sin\theta_j \approx \theta_j$ and $\cos\theta_j \approx 1$. **[A3]:** The *coupling strength* in (1) is given by $\mu := m/M$. **[A4]**: the damping and the nonlinearity of the van der Pol term in the pendulums is small, i.e. $\delta\eta\omega^{-1} = \mu p$ and $\nu\eta\omega^{-1} = \mu c$, with $p, c \in \mathbb{R}_+$. Finally, in order to get a dimensionless model, the following definitions are considered: $\tau = \sqrt{g/l}$ and $y_j = x_j/l$. Under these assumptions and considerations, system (1) can be written as a weakly nonlinear system of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mu\Phi(x), \tag{2}$$

where the overdot denotes differentiation with respect to τ, $\mathbf{x} = (\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, y_1, \dot{y}_1, y_2, \dot{y}_2)^T$, $\mu = \frac{m}{M}$ is the *coupling strength* , and

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2q & 2s & -q & -s \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -q & -s & q & s \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2q & -2s & q & s \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & q & s & -q & -s \end{pmatrix}, \tag{3}$$

where $q = \frac{k}{M\omega^2}$, $s = \frac{b}{M\omega}$, and the vector $\mathbf{\Phi}(\mathbf{x})$ is given by

$$\mathbf{\Phi}(\mathbf{x}) = (0, f_1 - g_1, 0, f_2 - g_2, 0, g_1, 0, g_2)^T, \tag{4}$$

with $f_j = [c(\gamma^2 - \theta_j^2) - p]\dot{\theta}_j$, $g_j = \theta_j(1 + \dot{\theta}_j^2)$, for $j = 1, 2$. Finally, note that in (2), higher order terms in μ ($\mu \geq 2$) have been neglected.

Next, by using the transformation $\mathbf{x} = \mathbf{V}\mathbf{z}$, where $\mathbf{V}$ is the matrix of eigenvectors associated to matrix $\mathbf{A}$, see Eq. (3), and $\mathbf{z} := (z_1, \ldots, z_8)^T$, it is possible to transform system (2) to the diagonal form

$$\dot{\mathbf{z}} = \mathbf{\Lambda}\mathbf{z} + \mu\mathbf{V}^{-1}\mathbf{\Phi}(\mathbf{V}\mathbf{z}), \tag{5}$$

where $\mathbf{\Lambda}$ is the matrix of eigenvalues of matrix $\mathbf{A}$, see Eq. (3), and is given by $\mathbf{\Lambda} = diag(i, -i, i, -i, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$, in which i is the imaginary unit and the variables σ_l are defined by $\sigma_l = -a_l + ib_l$ with $a_l \in \mathbb{R}_+$ and $b_l \in \mathbb{R}$, for $l = 1, \ldots, 4$. Explicit expression for a_l and b_l are not provided because are too lengthy and are not important for the upcoming analysis.

For $\mu = 0$, the solutions of system (5) are

$$z_r(\tau) = \begin{cases} \alpha_r e^{\frac{2\pi}{T} i\tau}, & r = 1, 3, \\ \alpha_r e^{-\frac{2\pi}{T} i\tau}, & r = 2, 4, \\ \alpha_r e^{\sigma_{r-4}\tau}, & r = 5, 6, 7, 8 \end{cases} \tag{6}$$

where $T = 2\pi$ is the dimensionless period. Recall that the real part of σ_l, $l = 1, \ldots, 4$ is negative. Therefore, solutions z_r, $r = 5, \ldots, 8$ damp out, i.e. $\lim_{\tau\to\infty} z_r = 0$, $r = 5, \ldots, 8$. Therefore, the problem reduces to finding the values of α_r, $r = 1, \ldots, 4$, such that the solutions of (2) asymptotically approach to solutions (6). However, for $0 < \mu << 1$, the period of the solutions of system (2) will be slightly different from the period of solutions (6). In particular, it will be assumed that the period of the solutions of system (2) is of the form

$$\tilde{T} = T + \mu\tau_c = 2\pi + \mu\tau_c, \tag{7}$$

where τ_c is a correction for the 'unperturbed' period T (i.e. when $\mu = 0$), see (6).

In order to find an expression for α_r, $r = 1, \ldots, 4$, it is useful to apply the transformation used before, i.e. $\mathbf{x} = \mathbf{V}\mathbf{z}$, with $\mathbf{z} = (z_1, \ldots, z_8)^T$, and z_r as given in (6) for $r = 1, \ldots, 4$, and $z_r = 0$, for $r = 5, \ldots, 8$. This yields the following solutions for the angular displacements of the pendulums

$$\theta_j = -i\alpha_{2j-1}e^{i\tau} + i\alpha_{2j}e^{-i\tau}, \quad \text{for } j = 1, 2, \tag{8}$$

Due to the fact that the coupling is asymmetric, the amplitudes of θ_1 and θ_2 are expected to be different and with a phase difference. Consequently, the $\alpha_r's$ in (8) are assumed to have the form: $\alpha_1 = \alpha\upsilon e^{i\phi}$, $\alpha_2 = \alpha\upsilon e^{-i\phi}$, and $\alpha_3 = \alpha_4 = \alpha$, with α, ϕ, and υ, to be determined. Substituting this in (8) yields

$$\theta_1 = 2\upsilon\alpha\sin\left(\frac{2\pi}{\tilde{T}}\tau + \phi\right), \qquad \theta_2 = 2\alpha\sin\left(\frac{2\pi}{\tilde{T}}\tau\right). \tag{9}$$

Summarizing, we are looking for synchronous solutions of system (1) in the form of (9), i.e. solutions that have the same frequency $f := 1/\tilde{T}$, but different amplitude, indicated by υ and a constant phase shift ϕ.

3.1. Amplitude, phase, and period of the synchronous solutions

In order to find the values of α, υ, ϕ, and $\tilde{T}$ of solutions (9), we use Theorem 1, given in the Appendix. The corresponding periodicity conditions, see (16) in Theorem 1, are given by

$$\begin{aligned} Q_1 &= -\alpha^2\pi\left\{\upsilon e^{i\phi}\left[2\varepsilon - c\alpha^2\left(\upsilon^2 + 1\right)\right] + i\Psi_1\right\} = 0, \\ Q_2 &= \alpha^2\pi\left\{\upsilon e^{-i\phi}c\alpha^2\left(\upsilon^2 - 1\right) + i\Psi_2\right\} = 0, \\ Q_3 &= -2\alpha^2\pi\left\{\vartheta_1 + \vartheta_2\alpha^2 + |\iota_2|(\upsilon + \upsilon^3\alpha^2)\sin(\phi + \angle\iota_2)\right\} = 0, \end{aligned} \tag{10}$$

where $\varepsilon = c\gamma^2 - p$, $\vartheta_1 = \iota_4 + \varepsilon$, $\vartheta_2 = \iota_4 - c$, $\iota_4 = \frac{\iota_3 - \bar{\iota}_3}{i2}$, $\varsigma_2 = 1 + \alpha^2$, and

$$\begin{aligned}\Psi_1 &= \left(\bar{\iota}_2 \upsilon e^{-i\phi} - a\iota_1\right)\varsigma_1 + \left(\bar{\iota}_3 \upsilon e^{i\phi} - \iota_2\right)\varsigma_2,\\ \Psi_2 &= \left(\bar{\iota}_2 \upsilon e^{-i\phi} - a\bar{\iota}_1\right)\bar{\varsigma}_1 + \left(\bar{\iota}_3 \upsilon e^{-i\phi} - \bar{\iota}_2\right)\varsigma_2,\end{aligned} \quad (11)$$

where $\bar{\varsigma}_1$ is the conjugated term for $\varsigma_1 = \upsilon e^{i\phi}(1 + \upsilon^2\alpha^2)$, and $\bar{\iota}_l$ are the conjugated values for ι_l $(l = 1, 2, 3)$ which are defined by

$$\iota_1 = \frac{a_1 + is}{n_1 + in_2},\ \iota_2 = \frac{q + is}{n_1 + in_2},\ \iota_3 = \frac{a_2 + i2s}{n_1 + in_2}, \quad (12)$$

where $a_1 = q - 1$, $a_2 = 2q - 1$, $n_1 = 1 + q^2 - s^2 - 3q$, $n_2 = 2qs - 3s$.

The next step is to solve (10) in terms of the unknown variables α, υ, and ϕ corresponding to solutions (9).

Solving Q_3 in (10) in terms of the unknown variable α, yields

$$\alpha = \pm\sqrt{\frac{\varepsilon + \iota_4 + \upsilon|\iota_2|\sin(\phi + \angle\iota_2)}{c - \iota_4 - \upsilon^3|\iota_2|\sin(\phi + \angle\iota_2)}}, \quad (13)$$

where $|\cdot|$ and $\angle$ denote the absolute value and angle, respectively.

Now, the problem is to find υ and ϕ. This can be done by substituting (13) in the first two equations of (10) and solving these equations in terms of υ and ϕ. However, it is quite hard to obtain an analytic solution. Therefore, in this study, the unknowns υ and ϕ are numerically obtained by solving

$$Q_j(\upsilon, \phi) = 0 \quad j = 1, 2 \quad (14)$$

subject to the restriction that (13) is real.

On the other hand, the period $\tilde{T}$ of the synchronous solutions (9) is computed from (7) with τ_c as given in (20), see the Appendix. Hence, it follows that

$$\tau_c(\mu) = \mu\frac{P_4(\upsilon^*, \phi^*, \alpha^*)}{i\alpha^*}. \quad (15)$$

where υ^*, ϕ^*, and α^* are the solutions of (10) and $P_4(\upsilon^*, \phi^*, \alpha^*) = \pi\alpha\left\{\varepsilon - c\alpha^2 + i(\bar{\iota}_2\bar{\varsigma}_1 + \bar{\iota}_3\varsigma_2)\right\}$.

Finally, the stability of the synchronous solutions (9) can be investigated by looking at the roots of (19), see Theorem 1 in the Appendix.

4. Numerical results

In this section, a numerical study is presented. Equations (1) are numerically integrated by using the following parameter values: $M_j = 1.5$ [kg], $k_j = 500$ [N/m], $b_j = 0.0015$ [Ns/m], $m_j = 0.1$ [kg], $l_j = 0.125$ [m], $\delta_j = 0.001$ [Nms/rad], $\nu_j = 0.57$ [kg m^2/rad^{3}s], $\gamma_j = 0.07$ [rad], and $g = 9.81$ [m/s^2], for $j = 1, 2$. Some of these values have been obtained from [5]. The simulation time has been adjusted to 200 sec, and the Matlab function `ode45` has been used as the solver.

In a first study, the initial conditions are $\theta_1 = 0.02$ [rad], $\theta_2 = -0.03$ [rad], and the remaining conditions are set to zero. The obtained results are shown in the Figure 2. Panel (a) shows the complete time series corresponding to the angular displacements θ_1 and θ_2, whereas panel (b) depicts a snapshot of the 'steady' behavior. Clearly, the pendulums oscillate almost in anti-phase but with different amplitudes. The horizontal dotted lines indicate the predicted amplitude, see (13) and (14), whereas the vertical dotted lines correspond to the predicted period from (7)-(15). It is clear to see that the theoretical and numerical results are in good agreement. In fact, the relative errors between numerical and teoretical results are as follows: 0.47% and 2.52% for υ and α, respectively, 3.63% for the period $\tilde{T}$, and 0.71% for the phase ϕ. Finally, the roots of (19) for this example are $\chi_1 = -3.25 - i0.61$, $\chi_2 = -0.45 + i5.67$, and $\chi_3 = -0.6 - i4.47$. Therefore, the obtained solution is stable.

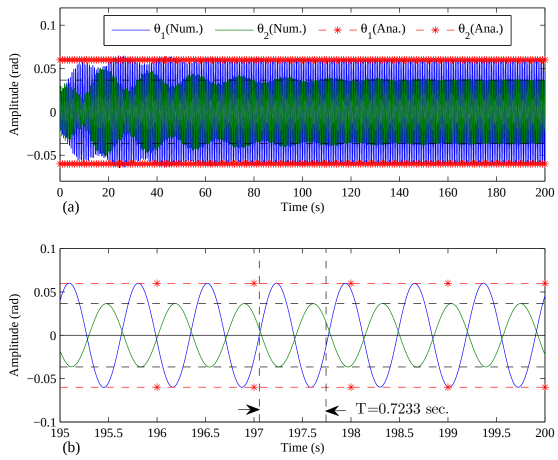


Figure 2: The pendulums exhibit frequency synchronization, with a phase shift close to π (anti-phase), but with different amplitudes.

In a second study, the initial conditions are $\theta_1 = 0.02$ [rad] and $\theta_2 = 0.03$ [rad]. The obtained results are shown in Figure 3. Again, the top panel shows the complete time series and the bottom panel shows a snapshot of the steady behavior. In this case the pendulums almost synchronize in in-phase. However, the pendulum at the top platform (red line), has a larger amplitude than the pendulum at the lower platform (blue line). Again, the predicted amplitudes and period are denoted by the horizontal and vertical dotted lines, respectively. The relative errors between numerical and theoretical results are 2.53% and 2.94% for υ and α, respectively, 2.36% for the period $\tilde{T}$ and 16.99% for the phase ϕ. About the stability, the roots of (19) are $\chi_1 = -5.94 + i0.97$, $\chi_2 = -0.1 + i4.17$, and $\chi_3 = -0.72 - i4.55$. Hence, the obtained solution is stable.

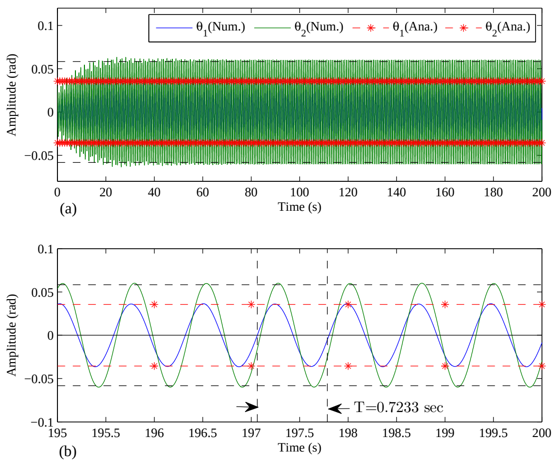


Figure 3: The pendulums exhibit frequency synchronization, with a phase shift close to 0 (in-phase), but with different amplitudes.

5. Conclusions

A novel coupling structure, in which a pair of pendulums are asymmetrically coupled has been presented. The lack of symmetry in the coupling results in loss of complete synchrony. However, it is still possible to observe frequency synchronization, with a constant phase shift, either close to zero or to π. However, further research of the asymmetric coupling presented here is necessary. For example, a study of the limit behavior of the coupled pendulums as a function of the parameters of the coupling, e.g. coupling strength, stiffness, and damping, seems in order, cf. [8]. Likewise, an experimental study of the results presented here is still missing.

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Appendix

Theorem 1 ([6, 7]) *Periodic solutions with period $T^*(\mu) = T + \tau_c(\mu)$ for the autonomous system (5), becoming at $\mu = 0$ periodic (period T) solutions (6) of the fundamental system, i.e. system (5) with $\mu = 0$, can correspond only to such values of constants $\alpha_1, \ldots, \alpha_{k-2}, \alpha_{k-1} = \alpha_k$, which satisfy equations*

$$Q_r(\alpha_1, \ldots, \alpha_k) := \alpha_k n_k P_r - \alpha_r n_r P_k = 0, \qquad r = 1, \ldots, k-1, \tag{16}$$

where

$$P_r(\alpha_1, \ldots, \alpha_k) = \int_0^T f_r(y_1^0, \ldots, y_l^0) e^{-in_r\omega t} dt \tag{17}$$

$$= \int_0^T f_r(\alpha_1 e^{in_1\omega t}, \ldots, \alpha_k e^{in_k\omega t}, 0, \ldots, 0) e^{-in_r\omega t} dt, \quad r = 1, \ldots, k. \tag{18}$$

If for a certain set of constants $\alpha_1 = \alpha_1^*, \ldots, \alpha_{k-2} = \alpha_{k-2}^*, \alpha_{k-1} = \alpha_k = \alpha_k^*$ which satisfy equations (16), the real parts of all roots χ of the following characteristic equation are negative [1]

$$p(\chi) = det\left(\left.\frac{\partial Q}{\partial \alpha}\right|_{\alpha=\alpha^*} - \alpha_k^* n_k \chi I \right) = 0, \tag{19}$$

then, for sufficiently small μ, this set of constants will indeed correspond to a unique, analytically w.r.t. μ, stable periodic solution of system (5) with period $T^*(\mu) = T + \tau_c(\mu)$. If the real part of at least one root of equation (19) is positive, then the corresponding solution is unstable. With accuracy up to terms of order μ, the period correction $\tau_c(\mu)$ is determined by

$$\tau_c(\mu) = -\mu \frac{P_k(\alpha_1^*, \ldots, \alpha_{k-2}^*, \alpha_k^*, \alpha_k^*)}{\lambda_k \alpha_k^*}. \tag{20}$$

[1] $\left.\frac{\partial Q}{\partial \alpha}\right|_{\alpha=\alpha^*} = \left.\begin{bmatrix} \frac{\partial Q_1}{\partial \alpha_1} & \cdots & \frac{\partial Q_1}{\partial \alpha_{k-1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial Q_{k-1}}{\partial \alpha_1} & \cdots & \frac{\partial Q_{k-1}}{\partial \alpha_{k-1}} \end{bmatrix}\right|_{\alpha_1=\alpha_1^*,\ldots,\alpha_{k-1}=\alpha_k^*}$ and $I \in \mathbb{R}^{k-1 \times k-1}$ is the identity matrix.