

Necessary and sufficient conditions for a 1-D DBCNN with an input to be stable in terms of connection coefficients

Hajime HARA[†] and Tetsuo NISHI[‡]

[†]Department of Electronics and Photonic Systems Engineering, Hiroshima Institute of Technology

2-1-1 Miyake, Saeki-ku, Hiroshima 731-5193 Japan

[‡]Faculty of Science and Engineering, Waseda University

3-14-9 Okubo, Shinjuku-ku, Tokyo 169-0072 Japan,

Email: hara@cc.it-hiroshima.ac.jp, nishi-t@waseda.jp

Abstract—We give necessary and sufficient conditions for a 1-D DBCNN (one-dimensional discrete-time binary cellular neural network) with an input to be stable in terms of connection coefficients. The results are complete generalization of our previous one[14], in which the input was assumed to be zero.

1. Introduction

One of the most fundamental problems of cellular neural networks (CNN) from the theoretical point of view is the stability, while the ability for 1-dimensional and 2-dimensional signal processing is important from the practical point of view [1]-[6]. However many of stability problems still remain unsolved completely even for one-dimensional CNN's[7]-[12].

In this paper we study on the stability of a one-dimensional discrete-time binary cellular neural networks (abbreviated as a 1-D DBCNN) with an input. Though the 1-D DBCNN is the simplest 1-D system, its stability problem is not so easy to solve. Previously we gave the necessary and sufficient conditions for the 1-D DBCNN to be stable in terms of “changeable sets”, which is briefly explained later [11][12]. But the relation between the conditions in terms of changeable set and connection coefficients is not explicit. So we recently investigated the stability conditions in terms of connection coefficients for the case of no input 1-D DBCNN [13][14]. In this paper we give the necessary and sufficient conditions for stability in terms of connection coefficients for a general 1-D DBCNN “with an input”. The discussion heavily depends on the the results in [12].

2. Preliminaries

The behavior of a 1-D DBCNN denoted by S can be described by the equation:

$$x(k+1) = \text{sgn}[Ax(k) + Bu + \theta\mathbf{1}] \quad (1)$$

where $x(k) = [x_1(k), \dots, x_n(k)]^T$ and $u = [u_1, \dots, u_n]^T$ are respectively a binary state vector at time k

and a binary time-invariant input vector, n is the dimension of S , A and B are $n \times n$ matrices determined by the A- and B-templates, θ is a scalar representing the threshold value, and $\mathbf{1}$ is an n -dimensional column vector consisting of 1 only¹. In particular $x(0)$ is an initial state vector, which can be used as another input data in many applications. We assume a 1-neighborhood DBCNN. Then Eq.(1) can be rewritten in a scalar form as:

$$\begin{aligned} x_i(k+1) &= \text{sgn} [\beta x_{i-1}(k) + \alpha x_i(k) + \gamma x_{i+1}(k) \\ &+ \hat{\beta} u_{i-1} + \hat{\alpha} u_i + \hat{\gamma} u_{i+1} + \theta], \\ &(i = 1, 2, \dots, n; k = 0, 1, 2, \dots) \quad (2) \end{aligned}$$

When we calculate $x_i(k+1)$ by Eq.(2), we have to define the boundary values $x_0(k)$ and $x_{n+1}(k)$ for the state vector x and u_0 and u_{n+1} for the input vector u , respectively. The fixed boundary considered in this paper means that $x_0(k)$ and $x_{n+1}(k)$ are constants independent of k .

Definition 1: A 1-D DBCNN S is said to be stable, if no limit cycle occur for any $x(0)$, any u , any boundary conditions on x and u , **and any value of the dimension n** . The 1-D DBCNN being not stable are said to be unstable.

When we discuss the stability of a 1-D DBCNN, we can formulate the problem in two ways as follows:

Problem 1: Prescribed parameters α , β and γ , can we determine the parameters $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and θ so that S to be stable for any input data u ?

Problem 2: Prescribed parameters α , β , γ , $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and θ , is the system S stable for any input data u ?

In this paper we give the answer to the above two problems. In particular, we show that the stability conditions for Problem 1 are essentially the same as those for no input case.

3. Summary of previous results

In this section we give the summary of the stability conditions for a 1-D DBCNN.

¹In the case of cellular automata the “sgn” function in Eq.(1) should be replaced with an arbitrary logic function.

3.1. General stability conditions in terms of changeable sets

3.1.1. Case where $\hat{\alpha} = \hat{\beta} = \hat{\gamma} = 0$ (no input case)

Then S can be described as follows:

$$x_i(k+1) = \text{sgn} [\beta x_{i-1}(k) + \alpha x_i(k) + \gamma x_{i+1}(k) + \theta] \quad (3)$$

The triple $(x_{i-1}(k), x_i(k), x_{i+1}(k))$ takes one of the following eight patterns; $(-, -, -)$, $(-, -, +)$, $(-, +, -)$, $(-, +, +)$, $(+, -, -)$, $(+, -, +)$, $(+, +, -)$, and $(+, +, +)$ where “+” and “-” mean +1 and -1, respectively. For some of these eight triples, $x_i(k+1)$ changes from $x_i(k)$ and for other triples $x_i(k+1)$ is the same as $x_i(k)$. We call the former triples “changeable patterns” for the prescribed parameters α , β , γ and θ and the latter ones “invariant patterns”. We denote the set of all changeable patterns by Φ , which is called a changeable set.

Throughout this paper the variables y_i and y'_i ($i = 1, 2, \dots$) denote the binary values 1 or -1 and “ \bar{y}_i ” means

$$\bar{y}_i = \begin{cases} 1 & \text{if } y_i = -1 \\ -1 & \text{if } y_i = 1 \end{cases} \quad (4)$$

Then we have:

Theorem 1[12]: The system S described by Eq.(3) is unstable if and only if at least one of the following two conditions (Eqs.(5)- (6)) holds for some y_i .

$$\Phi \supseteq \{(y_1, y_2, y_3), (y_1, \bar{y}_2, y_3)\} \quad (5)$$

$$\Phi \supseteq \{(y_1, y_2, y_3), (y_1, \bar{y}_2, \bar{y}_3), (\bar{y}_1, y_2, \bar{y}_3)\} \quad (6)$$

3.1.2. General 1-D DBCNN (nonzero input case)

Then Eq.(2) can be written as:

$$x_i(k+1) = \text{sgn} [\beta x_{i-1}(k) + \alpha x_i(k) + \gamma x_{i+1}(k) + \theta_i] \quad (7)$$

$$\theta_i = \hat{\beta}u_{i-1} + \hat{\alpha}u_i + \hat{\gamma}u_{i+1} + \theta, \quad (i = 1, 2, \dots, n). \quad (8)$$

Note that this general case corresponds to varying threshold case and that θ_i takes one of eight values, i.e., $\pm\hat{\beta}\pm\hat{\alpha}\pm\hat{\gamma}$, which we denote as θ_{0j} ($j = 1, 2, \dots, 8$). Note that θ_{0j} ($j = 1, \dots, 8$) are determined from the parameters, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, and θ , but that θ_i ($i = 1, 2, \dots, n$) depends not only the above parameters but also u . So each θ_{0j} has the corresponding changeable set denoted by $\Phi(\theta_{0j})$ as follows:

Since $x_{i-1}(k)$, $x_i(k)$ and $x_{i+1}(k)$ take 1 or -1, $\beta x_{i-1}(k) + \alpha x_i(k) + \gamma x_{i+1}(k)$ takes one of eight values, $\pm\beta \pm\alpha \pm\gamma$. We classify $\pm\beta \pm\alpha \pm\gamma$ into two classes; $\pm\beta +\alpha \pm\gamma$ are called α -terms and $\pm\beta -\alpha \pm\gamma$ are called $\bar{\alpha}$ -terms. Of course the value α may be positive or negative. For example, $\beta +\alpha -\gamma$ and $-\beta +\alpha -\gamma$ are α -terms and $\beta -\alpha -\gamma$ and $-\beta -\alpha -\gamma$ are $\bar{\alpha}$ -terms.

If one of α -terms, for example, $\beta +\alpha -\gamma$ satisfies $\beta +\alpha -\gamma > -\theta_{0j_0}$, then we have $\beta +\alpha -\gamma +\theta_{0j_0} > 0$. This means from Eq.(7) that if $\theta_i = \theta_{0j_0}$ and if

$(x_{i-1}(k), x_i(k), x_{i+1}(k)) = (+, +, -)$, then $x_i(k+1)$ is the same as $x_i(k)$. Thus the triple (x_{i-1}, x_i, x_{i+1}) ($(+, +, -)$ in the above example) corresponding to the α -term greater than $-\theta_{0j_0}$ is not a changeable pattern and therefore is not contained in the changeable set $\Phi(\theta_{0j_0})$ corresponding to θ_{0j_0} , i.e., $(+, +, -) \notin \Phi(\theta_{0j_0})$.

Conversely if $\beta +\alpha -\gamma < -\theta_{0j_0}$, then the triple $(+, +, -)$ is a changeable pattern and is therefore contained in $\Phi(\theta_{0j_0})$, i.e., $(+, +, -) \in \Phi(\theta_{0j_0})$ ². Of course $\Phi(\theta_{0j_0})$ may be the null set.

We have similar results for $\bar{\alpha}$ -terms. That is, if $\beta -\alpha -\gamma > -\theta_{0j_0}$, then $(+, -, -) \in \Phi(\theta_{0j_0})$ and if $\beta -\alpha -\gamma < -\theta_{0j_0}$, then $(+, -, -) \notin \Phi(\theta_{0j_0})$.

Summarizing the above, we have:

Lemma 1: To each θ_{0j} there exists the corresponding changeable set $\Phi(\theta_{0j})$, which consists of changeable patterns corresponding to α -terms less than $-\theta_{0j}$ and those corresponding to $\bar{\alpha}$ -terms greater than $-\theta_{0j}$.

We will next define the S·T-point, the S-point, and the T-point, which are very important to state our stability conditions.

Definition 2: If a changeable set $\Phi(\theta_{0j})$ includes (y_1, y_2, y_3) and (y_1, \bar{y}_2, y_4) (resp. (y_1, y_2, y_3) and (y_4, \bar{y}_2, y_3)) for some $y_i = \pm 1$, then we say that θ_{0j} is a starting point or simply an S-point (resp., a terminal point or simply a T-point). Similarly if the changeable set $\Phi(\theta_{0j})$ includes (y_1, y_2, y_3) and (y_1, \bar{y}_2, y_3) , we say that θ_{0j} or $\Phi(\theta_{0j})$ is a S·T-point.

The S·T-point is an S-point as well as a T-point, but a point being both an S-point and T-point is not necessarily an S·T-point.

As easily seen, we cannot assign arbitrary values to all θ_{0j} ($j = 1, \dots, 8$). In order to represent the feasible θ_{0j} ($j = 1, \dots, 8$), we define a directed graph $G = G(V, E)$ as follows: Let u'_i ($i = 1, 2, 3, 4$) be ± 1 . Then the vertices V is a set of all triples (u'_1, u'_2, u'_3) , and two edges starting at the vertex (u'_1, u'_2, u'_3) are connected to the vertices (u'_2, u'_3, u'_4) $u'_4 = \pm 1$. We call this graph “the transition graph” (see Fig. 1 in [12]).

Lemma 2: Suppose that for some θ_{0j} $\Phi(\theta_{0j})$ contains both (y_1, y_2, y_3) and (y_1, \bar{y}_2, y_3) for some $y_i \in \{1, -1\}$ ($i = 1, 2, 3$). Then the system S has a limit cycle with $n = 1$, i.e., if there exist an S·T-point, then there exist a limit cycle with the dimension $n = 1$.

Definition 3: In the transition graph we call a directed path starting from an S-point to a T-point an S-T path. An S-T path without a loop is called a simple S-T path. A point on P being neither S-point nor T-point is called an intermediate point or shortly an I-point.

Theorem 2: The 1-D DBCNN S with an input is unstable under unspecified fixed boundaries if and only

²In this paper we exclude the case where $\pm\beta \pm\alpha \pm\gamma = 0$, which is a pathological case from the practical point of signal processing.

if the graph corresponding to the transition graph of the input has either an S-T-point or an S-T path.

3.1.3. Stability conditions for 1-D DBCNN without input

We will omit the results for this case, because they are essentially the same as a part of those in Section 4.

4. Stability conditions in terms of connection coefficients for general 1-D DBCNN cases

Theorems 1 and 2 states the necessary and sufficient conditions in terms of “changeable set”. In this section we derive direct conditions in terms of connection coefficients. For this purpose we have to clarify the relation between the connection coefficients and the changeable set. Then $\Phi(\theta_{0j_0})$ can be obtained from Lemma 1 by comparing the values of $\pm\beta\pm\alpha\pm\gamma$ with $-\theta_{0j_0}$ defined above. This is easily done by drawing figures shown in Tables 3 and 5.

The α -terms are represented as $y_1\beta+\alpha+y_2\gamma$. When we arrange them in the order of value, the largest (resp. smallest) value is apparently $|\beta|+\alpha+|\gamma|$ (resp. $-|\beta|+\alpha-|\gamma|$) and the second largest is $|\beta|+\alpha-|\gamma|$ or $-|\beta|+\alpha+|\gamma|$. Without loss of generality we assume that

$$|\beta|+\alpha-|\gamma| \geq -|\beta|+\alpha+|\gamma|. \quad (9)$$

i.e.,

$$|\beta| \geq |\gamma|. \quad (10)$$

Then α - and $\bar{\alpha}$ -terms are respectively arranged in the order of values as in two columns in Table 1.

$\bar{\alpha}$ -terms	α -terms
$ \beta -\alpha+ \gamma $	$ \beta +\alpha+ \gamma $
$ \beta -\alpha- \gamma $	$ \beta +\alpha- \gamma $
$- \beta -\alpha+ \gamma $	$- \beta +\alpha+ \gamma $
$- \beta -\alpha- \gamma $	$- \beta +\alpha- \gamma $

Table 1

where the relative relation between $\bar{\alpha}$ -terms and α -terms should be changed by the value of α .

The system S include an S-T-point if and only if $\Phi(\theta_{0j})$ contains both $y_1|\beta|+\alpha+y_2|\gamma|$ and $y_1|\beta|-\alpha+y_2|\gamma|$.

To show an S-point and a T-point explicitly, we mark each term in Table 1 as in Table 2:

$\bar{\alpha}$ -terms		α -terms	
1	$ \beta -\alpha+ \gamma $	5	$ \beta +\alpha+ \gamma $
2	$ \beta -\alpha- \gamma $	6	$ \beta +\alpha- \gamma $
3	$- \beta -\alpha+ \gamma $	7	$- \beta +\alpha+ \gamma $
4	$- \beta -\alpha- \gamma $	8	$- \beta +\alpha- \gamma $

Table 2

We see that two terms with the same number in the left (resp. right) hand side of each column of Table 2 represent an S-point (resp. T-point). For example, if $\Phi(\theta_{0j})$ contains $|\beta|-\alpha+|\gamma|$ and $|\beta|+\alpha+|\gamma|$, which have the same number “2” in the left hand of $\bar{\alpha}$ - and α -columns, then S include an S-point.

To determine $\Phi(\theta_{0j})$ is easily done by drawing more exactly drawing the relation of the value of terms in Table 2. We have to consider several cases as follows: Case 1: $-|\beta|+\alpha-|\gamma| > |\beta|-\alpha+|\gamma|$ i.e., $\alpha > |\beta|+|\gamma|$

In this case we cannot choose θ_{0j} (i.e., $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, and θ) such that there arises either an S-T-point or an S-point and a T-point. So

Lemma 3:the system S is stable independently of the values $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, and θ in Case 1.

Case 2: $|\beta|-\alpha+|\gamma| > |\beta|+\alpha-|\gamma| > |\beta|-\alpha-|\gamma| > -|\beta|+\alpha+|\gamma|$ i.e., $|\gamma| > \alpha > 0$ and $|\beta| - |\gamma| > \alpha$

In this case we have Table 3 as follows:

$\bar{\alpha}$ -terms		α -terms					
...	A			
...	...	2	$ \beta +\alpha+ \gamma $	7	B		
1	$ \beta -\alpha+ \gamma $	5	C		
...	1	$ \beta +\alpha- \gamma $	8	D	
2	$ \beta -\alpha- \gamma $	6	E		
...	4	$- \beta +\alpha+ \gamma $	5	F	
...	G	
3	$- \beta -\alpha+ \gamma $	7	...	3	$- \beta +\alpha- \gamma $	6	H
...	I
4	$- \beta -\alpha- \gamma $	8
...

Table 3

From Table 3 and the definition of the S-T-point, S-point, and T-point, we learn the existence of them as follows:

	S-point	T-point	S-T-point
A			
B			
C	exist	exist	
D		exist	
E		exist	
F		exist	
G	exist	exist	
H			
I			

Table 4

Lemma 4:The system S can be unstable by choosing the parameters so that at least one of θ_{0j_0} lies as an S-points in the domain C and G in Table 3 and others in the domain C-G.

Case 3: $|\beta|+\alpha+|\gamma| > |\beta|-\alpha-|\gamma| > |\beta|+\alpha-|\gamma| > -|\beta|-\alpha+|\gamma|$ i.e., $|\gamma| > -\alpha > 0$ and $|\beta| - |\gamma| > -\alpha$

In this case we have Tables 5 and 6 in a similar way as follows:

$\bar{\alpha}$ -terms			α -terms		
...	A
1	$ \beta - \alpha + \gamma $	5	B
...	2	$ \beta + \alpha + \gamma $	7
...	C
2	$ \beta - \alpha - \gamma $	6	D
...	1	$ \beta + \alpha - \gamma $	8
...	E
3	$- \beta - \alpha + \gamma $	7	F
...	4	$- \beta + \alpha + \gamma $	5
...	G
4	$- \beta - \alpha - \gamma $	8	H
...	3	$- \beta + \alpha - \gamma $	6
...	I

Table 5

	S-point	T-point	S·T-point
A			
B	exist	exist	exist
C	exist	exist	
D	exist	exist	exist
E		exist	
F	exist	exist	exist
G	exist	exist	
H	exist	exist	exist
I			

Table 6

In this case we have a similar result as Lemma 4. Case 4: $|\beta| - \alpha - |\gamma| > |\beta| + \alpha + |\gamma| > -|\beta| - \alpha + |\gamma|$ i.e., $-|\gamma| > \alpha > -|\beta|$ and $-|\beta| + |\gamma| > \alpha$

In this case we can get the tables corresponding to Tables 5 and 6, but will be omitted here because of no interest. In this case we can verify that S can be unstable if $|\beta| - \alpha - |\gamma| > \theta_{0j} > -|\beta| + \alpha - |\gamma|$.

Summarizing the results above, we have:

Theorem 3: For the Problem A, the conditions for stability on α , β , and γ is the same as those for no input case. For the Problem B, the stability conditions can easily be obtained from Tables 2, 4, and 6.

5. Conclusion

We gave the necessary and sufficient conditions for 1-D DBCNN with input to be stable.

Acknowledgment

The authors would like to express their gratitude to the reviewers for helpful suggestions. This research was supported in part by Core Research for Evolutional Science and Technology (CREST), and Japan Science and Technology Agency (JST) and by the Grants-in-Aid for Scientific Research (C) no. 16560339

from the Japanese Ministry of Education, Culture, Sports, Science and Technology.

References

- [1] L.O. Chua, L. Yang, "Cellular Neural Network: Theory and Applications", IEEE Trans. CAS, vol.35, no.10, pp.1257-1272, October 1988 and IEEE Trans. CAS, vol.35, no.10, pp.1273-1290, October 1988.
- [2] T. Roska, "CNN Software Library ver.7.2", The Computer and Automation Institute of the Hungarian Academy of Sciences, 1998.
- [3] A. Zarandy "The art of CNN template design", Int. J. Circ. Theor. Appl., vol. 27, no.1, pp. 5-23, Jan-Feb 1999.
- [4] Z.J. Lu, and D.R. Liu "A new synthesis procedure for a class of cellular neural networks with space-invariant cloning template", IEEE Trans. CAS-II vol.45, no.12, pp.1601-1605, Dec 1998.
- [5] I. Fajfar, F. Bratkovic, et al. "A rigorous design method for binary cellular neural networks", Int. J. Circ. Theor. Appl., vol. 26, no. 4, pp.365-373, Jul-Aug. 1998.
- [6] H. Harrer and J.A. Nossek "Discrete-time cellular neural networks", Int. J. Circ. Theor. Appl., vol. 20, no. 5, pp. 453-467, Sept-Oct. 1992
- [7] G. De Sandre "Stability of 1-D-CNN's with Dirichlet boundary conditions and global propagation dynamics", IEEE Trans. CAS-I vol.47, no.6, pp.785-792, June 2000.
- [8] G. Manganaro and J.P. de Gyvez "One-dimensional discrete-time CNN with multiplexed template-hardware", IEEE Trans. CAS-I, vol.47, no. 5, pp.764-769, May 2000.
- [9] P. Thiran, G. Setti, and M. Hasler "An approach to information propagation in 1-D cellular neural networks - Part I: Local diffusion", IEEE Trans. CAS-I vol.45, no. 8, pp.777-789, Aug 1998.
- [10] P.P. Civallesi a,d M. Gilli "Global dynamic behavior of a 3-cell connected component detector CNN", Int. J. Circ. Theor. Appl., vol. 23, no.2, pp.117-135, Mar-Apr 1995.
- [11] Tetsuo Nishi, Hidenori Sato, Norikazu Takahashi, "Necessary and Sufficient Conditions for One-Dimensional Discrete-Time Binary Cellular Neural Networks with Both A- and B-Templates to be Globally Stable", Proceedings of the IEEE International Symposium on Circuits and Systems, vol.1, pp.633-636, May 2002.
- [12] H. Sato, T. Nishi, and N. Takahashi, "Necessary and sufficient conditions for one-dimensional discrete-time binary cellular neural networks with unspecified fixed boundaries to be stable", Trans. IEICE, vol.E85-A, no.9, pp.2036-2043, 2002.
- [13] T. Nishi, H. Hara, and N. Takahashi, "Necessary and sufficient conditions for one-dimensional discrete-time binary cellular neural network to be stable—No input case—", Proc. The 8th IEEE International Biannual Workshop on Cellular Neural Networks and their Applications, Budapest, Hungary, July, 2004.
- [14] H. Hara, and N. Takahashi, T. Nishi, "Necessary and sufficient conditions for one-dimensional discrete-time binary cellular neural network with nonsymmetric connections to be stable", Proc. The 47th Midwest Symposium on Circuits and Systems, Hiroshima, Japan, July, 2004.