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Emergence of hyperchaos and synchronization in a simple autonomous discrete system

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Abstract—In this work, diffusive coupling of a simple autonomous discrete system resulting in emergence of hyperchaos is presented. The simple discrete system would never generate chaos by itself unless it is coupled as proposed in this work. Furthermore, in-phase or anti-phase synchronization of two coupled simple autonomous discrete systems within a complex network is achieved, where the systems maintain the hyperchaotic emergent dynamics. In order to corroborate that the emerging dynamics are hyperchaotic, we calculate the Lyapunov exponents.

1. Introduction

Synchronization of complex networks has received a great interest in different fields of science and technology. In particular, synchronization of complex dynamical networks with chaotic systems as nodes, see e.g. [1]-[2]. Interaction among coupled nodes within a complex network plays an important role in the emerging dynamics of networks, for example, emergence of chaos [3]. At first instance, there is the case of chaos or hyperchaos emergence with coupled chaotic oscillators in periodic regime, see for example [4]-[6]. On the other hand, there is the case of chaos or hyperchaos emergence of coupled non-chaotic systems [7]. The last case is the presented in this work, with the peculiarity that in addition to the emergence of hyperchaos, the coupled discrete systems achieve in-phase or anti-phase synchronization with a particular coupling configuration and parameters. This paper is organized as follows. In Section 2, some basic concepts on synchronization of complex dynamical networks are presented. In Section 3, equations of the autonomous discrete system are presented. In addition, the emergence of hyperchaos from interaction of only two simple autonomous discrete systems bidirectionally coupled is studied. In Section 4, we show synchronization of two bidirectionally coupled nodes, where in addition of emergence of hyperchaos, synchronization between the nodes is preserved. In Section 5, calculation of Lyapunov exponents that confirm hyperchaos emergence are presented. Finally, in Section 6 some conclusions are presented.

2. Complex networks

We consider a *complex network* composes of N identical nodes, linearly and diffusively coupled through the first state of each node. In this dynamical network, each node constitutes a *n-dimensional discrete-time map*. The state equations of this network are described by

$$\mathbf{x}_i(k+1) = f(\mathbf{x}_i(k)) + \mathbf{u_i}(k), \quad i = 1,2,\ldots,N, \quad (1)$$

where $\mathbf{x}_i(k) = (x_{i1}(k),\, x_{i2}(k),\, \ldots,\, x_{in}(k))^T \in \mathbb{R}^n$ are the state variables of the node i, $\mathbf{u_i}(k) = (u_{i1}(k), 0, \ldots, 0)^T \in \mathbb{R}^n$ is the input signal of the node i, and is defined by

$$\mathbf{u_i}(k) = c\sum_{j=1}^{N} a_{ij}\Gamma\mathbf{x}_j(k), \quad i = 1,2,\ldots,N, \quad (2)$$

the constant $c > 0$ represents the *coupling strength* of the complex network, and $\Gamma \in \mathbb{R}^{n\times n}$ is a constant 0-1 matrix linking coupled state variables. Whereas, $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{N\times N}$ is the *coupling matrix*, which represents the coupling topology of the complex network. If there is a connection between node i and node j, then $a_{ij} = 1$; otherwise, $a_{ij} = 0$ for $i \neq j$. The diagonal elements of coupling matrix $\mathbf{A}$ are defined as

$$a_{ii} = -\sum_{j=1,\ j\neq i}^{N} a_{ij} = -\sum_{j=1,\ j\neq i}^{N} a_{ji}, \quad i = 1,2,\ldots,N. \quad (3)$$

If the degree of node i is d_i, then $d_i = -a_{ii}$, $i = 1,2,\ldots,N$. Now, suppose that the complex network (1)-(2) is connected without isolated clusters. Then, $\mathbf{A}$ is a symmetric irreducible matrix. In this case, it can be shown that zero is an eigenvalue of $\mathbf{A}$ with multiplicity 1 and all the other eigenvalues of $\mathbf{A}$ are strictly negative, see [8] and [9]. In accordance with [9] for discrete systems, the complex network (1)-(2) is said to achieve (asymptotically) **synchronization** if:

$$\mathbf{x}_1(k) = \mathbf{x}_2(k) = \ldots = \mathbf{x}_N(k), \ \ as \ \ k \to \infty. \quad (4)$$

The diffusive coupling condition (3) guarantees that the synchronization state is a solution, $\mathbf{s}(k) \in \mathbb{R}^n$, of an isolated node, that is

$$\mathbf{s}(k+1) = f(\mathbf{s}(k)), \tag{5}$$

where $\mathbf{s}(k)$ can be an *equilibrium point*, a *periodic orbit* or, a *chaotic attractor*. Thus, stability of the synchronization state,

$$\mathbf{x}_1(k) = \mathbf{x}_2(k) = \ ... \ = \mathbf{x}_N(k) = \mathbf{s}(k), \tag{6}$$

of complex network (1)-(2) is determined by the dynamics of an isolated node, i.e. -function f and solution $\mathbf{s}(k)$-, the coupling strength c, the inner linking matrix Γ, and the coupling matrix $\mathbf{A}$.

3. The Simple autonomous discrete system, coupling, and the emergence of hyperchaos

This section describes the state equations, the coupling and the emergence of hyperchaos for two coupled simple autonomous discrete systems (nodes).

3.1. Uncoupled simple autonomous discrete systems

Dynamical networks with discrete simple autonomous discrete systems are used.

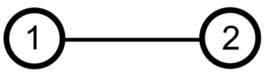

Figure 1: Scheme of a network with two bidirectionally coupled systems.

Figure 1 illustrates a simple network with two bidirectionally coupled autonomous discrete systems. State equations for this network are given as follows, the first node N_1 of the dynamical network, is described by

$$\begin{cases} w_1(k+1) &= w_1(k)b + u_{11}(k), \\ x_1(k+1) &= sin(w_1(k)), \end{cases} \tag{7}$$

with input signal

$$u_{11}(k) = c(\eta x_1(k) + x_2(k)), \tag{8}$$

the second node N_2 is given by

$$\begin{cases} w_2(k+1) &= w_2(k)b + u_{21}(k), \\ x_2(k+1) &= sin(w_2(k)), \end{cases} \tag{9}$$

with input signal

$$u_{21}(k) = c(\eta x_2(k) + x_1(k)), \tag{10}$$

Note that in the coupled nodes (7)-(10), two parameters η and b are introduced, these parameters directly affect the interaction of the autonomous discrete systems, therefore, appropriate choice of these parameters determines whether the emerging dynamic in the network is periodic, chaotic or hyperchaotic. Let us consider the following parameter value $b = 0.5$, and initial conditions: $w_1(0) = 2\pi 1$, $w_2(0) = 2\pi 1.1$, and $x_1(0) = x_2(0) = 0$. If coupling strength $c = 0$, then N_1 and N_2 are *uncoupled nodes*, i.e. $u_{11}(k) = u_{21}(k) = 0$; nodes N_1 and N_2 correspond to a discrete autonomous decreasing systems. Figure 2 shows state trajectories of $x_1(k)$ and $x_2(k)$ for isolated discrete systems (7)-(10) (due the nodes are discrete the default interpolation of Matlab is used, in order to better appreciate the temporary graphics). It can be seen that nodes N_1 and N_2 have decreasing behavior when they are uncoupled and under this scenario the isolated nodes in no way generate chaos.

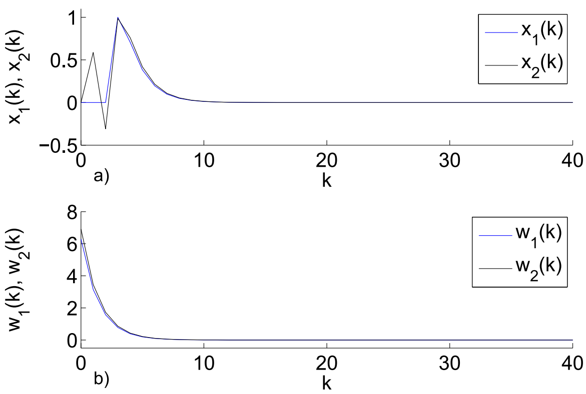

Figure 2: Temporal dynamics of states: a) $x_1(k)$ and $x_2(k)$, b) $w_1(k)$ and $w_2(k)$, with $u_{11}(k) = u_{21}(k) = 0$.

3.2. Emerging hyperchaos in the coupled autonomous discrete systems

On the other hand, if we use $\eta \neq 0$ and a coupling strength $c \neq 0$ in Eqs. (7)-(10), there is a complex emerging behavior in the network due to the interaction of the simple autonomous discrete systems, these dynamics are hyperchaotic and it will be demonstrated in a later section. Figures 3 and 4 show the bifurcation diagram of $x_1(k)$ with respect to parameters η and b respectively, where we can see that there are interesting and complex behaviors.

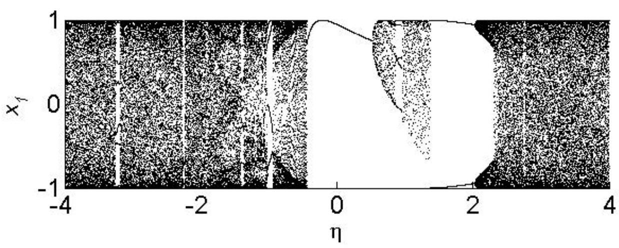

Figure 3: Bifurcation diagram of $x_1(k)$ with respect to η with $c = 1$ and $b = 0.5$.

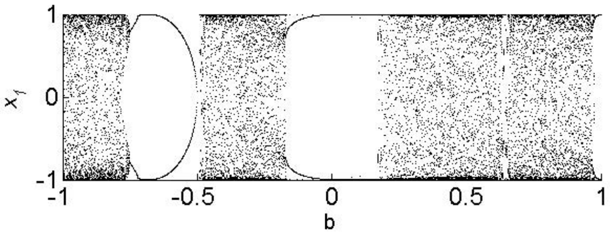

Figure 4: Bifurcation diagram of $x_1(k)$ with respect to b with $c = 1$ and $\eta = -0.5$.

Based on the analysis of the bifurcation diagrams, values of $c = 1$, $\eta = 3$, and $b = 0.5$ can be used to generate the Figure 5 that correspond to the temporal dynamics and attractors of the Eqs. (7)-(10).

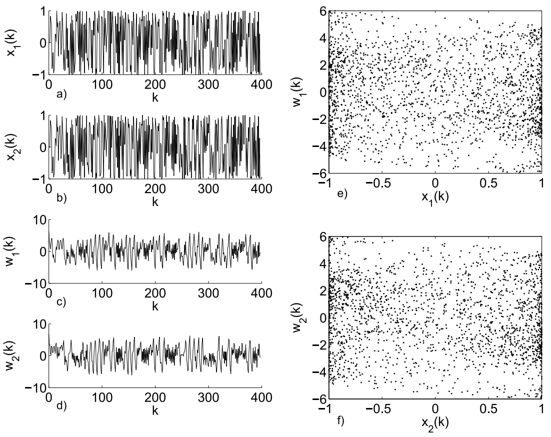

Figure 5: Time evolution: a) $x_1(k)$, b) $x_2(k)$, c) $w_1(k)$, d) $w_2(k)$, and attractors: e) $x_1(k)$ *vs* $w_1(k)$, f) $x_2(k)$ *vs* $w_2(k)$, with $c = 1$, $\eta = 3$ and $b = 0.5$.

In the next section, we will show that we can preserve the seemingly hyperchaotic dynamics and in addition the systems (7)-(10) achieve phase and anti-phase network synchronization.

4. Hyperchaotic network synchronization

In this section, synchronization of two bidirectionally coupled nodes N_1 and N_2, where the interactions still leads to emergence of seemingly hyperchaos dynamics is presented. Now consider the single network (7)-(10) with parameter values $c = 1$, $\eta = 1$, and $b = 0.5$, phase synchronization is achieved between nodes and the emergent behavior remains seemingly hyperchaotic as we can see in Figure 6 where state trajectories $x_1(k)$, $x_2(k)$, $w_1(k)$, $w_2(k)$, errors $x_1(k) - x_2(k)$, $w_1(k) - w_2(k)$, atractor $x_1(k)$ *vs* $w_2(k)$, and phase portrait $x_1(k)$ *vs* $w_2(k)$ are shown. We eliminated the first five iterations of the attractors in order to despise the transitory, so that, the presented simulation results can be seen in more detail.

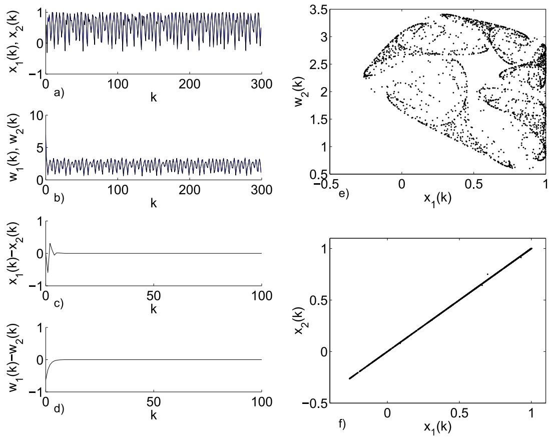

Figure 6: Time evolution: a) $x_1(k)$ (blue), $x_2(k)$ (black), b) $w_1(k)$ (blue), $w_2(k)$ (black), errors: c) $x_1(k) - x_2(k)$, d) $w_1(k) - w_2(k)$, attractor: $x_1(k)$ *vs* $w_2(k)$, and phase portrait: $x_1(k)$ *vs* $x_2(k)$, with $c = 1$, $\eta = 1$ and $b = 0.5$.

A numerical calculation of the synchronization was performed for $-10 \leq \eta \leq 10$ (for $k = 0, 1, ..., 2500$, so the calculation is approximated). After removing the first 2000 iterations of each state, we review $x_1(k) - x_2(k)$ (phase synchronization) and $x_1(k) + x_2(k)$ (anti-phase synchronization). If $|x_1(k) \pm x_2(k)| \geq 0.01$ (1% peak amplitude of $x_1(k)$), then, we establish no synchronization among nodes of the network (7)-(10). If $|x_1(k) - x_2(k)| < 0.01$, we establish phase synchronization and if $|x_1(k) + x_2(k)| < 0.01$ we establish anti-phase synchronization. Figure 7 shows a *synchronization diagram* of c with respect to η, where, if $\eta = \pm 1$, the network synchronizes for any value of $1 < c < -1$. Note that, with illustrative purposes only, c takes negative values.

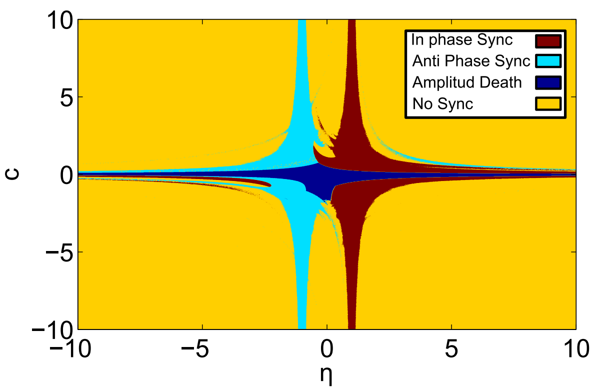

Figure 7: Hyperchaotic synchronization diagram for η *vs* c with $b = 0.5$.

In next section, we carried out the calculations of the Lyapunov exponents to verify the emergence of chaos and hyperchaos in the network (7)-(10).

5. Test for determining chaos and hyperchaos

Section 3 showed that for (7)-(10) with $c = 0$ in any way chaotic or hyperchaotic dynamics emerge. In this section, we verify if emerging dynamics are chaotic or hyperchaotic when $c \neq 0$, that is, due to the interaction of the simple autonomous discrete systems. To determine if emergent dynamics of network (7)-(10) are chaotic or hyperchaotic, the calculations of Lyapunov exponents are carried out, where one positive exponent indicates chaos, and two or more positive exponents indicate hyperchaos. For example, if we calculate the Lyapunov exponents of the network (7)-(10) with parameter values $c = 1$, $\eta = -2$, $b = 0.5$, we have, $L_1 = 0.3104$, $L_2 = 0.11702$, $L_3 = -0.23397$, $L_4 = -0.43666$, due that two positive Lyapunov exponents are present, the network (7)-(10) generates *hyperchaotic* dynamics. Various kinds of fractal dimensions can be estimated theoretically and empirically, as the Hausdorff dimension, Minkowski-Bouligand dimension, box-counting dimension, correlation dimension, Kaplan-Yorke dimension etc. see [10]. The Kaplan-Yorke dimension (calculated for a time series of 10000 iterations) for the proposed network is

$$D_{KY} = 3 + \frac{L_1 + L_2 + L_3}{|L_4|} = 3.443. \tag{11}$$

Both Lyapunov exponents and Kaplan-Yorke dimension were obtained by using the algorithm reported in [10]. In order to have a broad overview of the dynamics that the coupled systems (7)-(10) can generate, we performed the diagram of Figure 8, where we can see what values of c and η can generate chaos or hyperchaos.

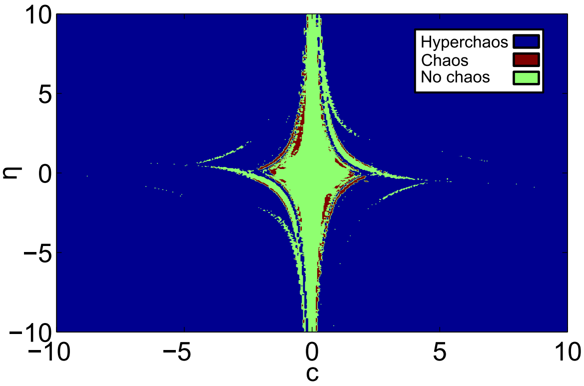

Figure 8: Diagram of chaos and hyperchaos emergence for η *vs* c with $b = 0.5$.

6. Conclusions

In this paper the emergence of chaos and hyperchaos in networks with discrete periodic oscillators, as we can noted in the carried out extensive numerical analysis, was presented. The hyperchaotic discrete coupled system can potentially be used in secure communications, where we could use microcontrollers, FPGAs, or any other device to the experimental implementation in the safely transmission of information. In future works, it is expected to perform more detailed research and theoretical analysis of the presented and larger networks.

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