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Synthesis of Nonautonomous Systems with Specified Limit Cycles

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Abstract—This paper deals with a synthesis of a nonautonomous system with a stable limit cycle. By extending Green's method, by which arbitrary periodic solutions can be designed in autonomous systems, it is shown that a nonautonomous system whose transient trajectories converge to a prescribed limit cycle can be synthesized. Furthermore, we apply receding horizon control to determine the optimal time-depending parameters in the system. The validity of the proposed synthesis is illustrated by examples of 3-dimensional systems.

1. Introduction

Limit cycles are one of the most important phenomena in nonlinear dynamical systems. As well as stability analysis of limit cycles, the inverse problem of synthesizing a nonlinear system with a specified stable limit cycle has been studied, and several methods for the inverse problem have been proposed [1, 2, 3]. The inverse problems have been applied in many engineering fields. For example, a periodic primitive motion of a humanoid robot can be regarded as a limit cycle in a sensory space [4].

The method proposed by Green [2] guarantees that all trajectories of $\dot{x} = f(x) + g(x)$ starting from any initial point converge to a specified limit cycle satisfying a constraint equation $V(x) = 0$. In the synthesized system, however, the rate of convergence and the transient trajectories are not explicitly taken into account. So some time-dependent parameters are introduced into the nonlinear system and we address control of transient trajectories for the desired convergence. For this purpose, we first clarify some properties of a nonautonomous system which has a specified stable limit cycle. Then, by applying receding horizon control, the optimal parameters (control inputs) are obtained.

This paper is organized as follows. Section 2 reviews some fundamental results reported in [2, 5]. Section 3 discusses an extension of Green's method in order to apply it to nonautonomous systems. In Section 4, receding horizon control is employed as a method for optimal control of the system.

2. Preliminaries

In this section, some underlying concepts used throughout this paper are presented. We first review a synthesis of an autonomous system with a specified limit cycle discussed in [2]. The specified limit cycle is assumed to satisfy a constraint equation $V(x) = 0$. The constraint equation is used for guaranteeing a stability property of the limit cycle of the system like a Lyapunov function.

Consider the following autonomous system:

$$\dot{x} = f(x) + g(x), \tag{1}$$

where $f : R^n \to R^n$ and $g : R^n \to R^n$ are continuously differentiable. Green provides the following theorem and corollary [2]:

Theorem 1 (Green [2]) If there exists a continuously differentiable function $V : D \to R^m$, where D is a subset of R^n and m is a positive integer less than n, such that

- $$\frac{\partial V(x)}{\partial x} f(x) \equiv 0 \quad \forall x \in D, \tag{2}$$
- For each μth component V_μ of V $(1 \le \mu \le m)$,
$$\frac{\partial V_\mu(x)}{\partial x} g(x) V_\mu(x) < 0 \ \forall x \in D \text{ s.t. } V_\mu(x(t)) \neq 0, \tag{3}$$

then, for any trajectory $x(t)$ of (1), $\lim_{t\to\infty} V(x(t)) = 0$.

Corollary 1 (Green [2]) *Let $V_a : R^n \to R^{m_a}, V_b : R^n \to R^{m_b}$, where $0 < m_a + m_b = m$. Assume Theorem 1 is applicable to V_a. If all solutions of (1) with initial conditions in D are bounded, then Theorem 1 is applicable to V_b with the conditions (2) and (3) relaxed to apply only to $x(t) \in D$ such that $V_a(x) = 0$.*

For a nonautonomous system given by

$$\dot{x}(t) = F(x, t), \tag{4}$$

the following lemma is well-known.

Lemma 1 (J.-J.Slotine [5]) *Suppose that there exists a scalar function $W(t, x)$ satisfies the following conditions:*
- *$W(t, x)$ is lower bounded.*
- *$\dot{W}(t, x)$ is negative semi-definite.*
- *$\dot{W}(t, x)$ is uniformly continuous in time.*

Then, for any trajectory $x(t)$ of Eq. (4), $\dot{W}(t, x) \to 0$ as $t \to \infty$.

3. Nonautonomous systems with limit cycles

We consider a nonautonomous system described by

$$\dot{x} = f(x) + g(t, x), \tag{5}$$

where $D \subset R^n$, $f(x) : D \to R^n$ is continuously differentiable, and $g(t, x) : [0, \infty) \times D \to R^n$ is locally Lipschitz in x and uniformly continuous in t. Then, applying Lemma 1, we extend Theorem 1 as follows:

Lemma 2 *Let $V : D \to R^m$ be a function such that $\partial V_\mu/\partial x$ is uniformly continuous for each μth element V_μ of $V(1 \le \mu \le m)$. Suppose V satisfies the following conditions:*

- $\dfrac{\partial V(x)}{\partial x} f(x) \equiv 0 \quad {}^\forall x \in D$ *and,* (6)
- *for each* $V_\mu(1 \le \mu \le m)$,
  $\dfrac{\partial V_\mu(x)}{\partial x} g(t, x) V_\mu(x) < 0 \ {}^\forall x \in D$ *s.t.* $V_\mu(x(t)) \neq 0.$ (7)

Then, for any trajectory of (5), $\lim_{t\to\infty} V(x(t)) = 0.$

From Lemma 2, we can show that the same property as Corollary 1 holds for the nonautonomous system (5).

In the following, we will discuss a synthesis method of a nonautonomous system with a specified limit cycle. A closed curve $\xi(t)$ in an N-dimensional space can be expressed by the Fourier series as follows:

$$\xi(t) = \sum_{k=0}^{\infty} \left[\alpha_k \cos(k\omega t) + \beta_k \sin(k\omega t)\right], \tag{8}$$

where $\alpha_k, \beta_k \in R^n$. Using Chebyshev polynomials $T_k(\cdot)$ and $U_k(\cdot)$, (8) is rewritten as

$$\begin{aligned}\xi(t) &= \sum_{k=0}^{\infty} \left[a_k T_k(\cos(\omega t)) + b_k \sin(\omega t) U_k(\cos(\omega t))\right] \\ &= F_1(\cos(\omega t)) + \sin(\omega t) F_2(\cos(\omega t)),\end{aligned} \tag{9}$$

where a_k and $b_k \in R^N$ depend on α_k and β_k, $F_1 : [-1, 1] \to R^N$, and $F_2 : [-1, 1] \to R^N$. For the limit cycle given by (9), let $f : R^{N+2} \to R^{N+2}$ and $g : R \times R^{N+2} \to R^{N+2}$ be as follows:

$$f(x) = \omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -\frac{\partial F_1(x_2)}{\partial x_2} - x_1 \frac{\partial F_2(x_2)}{\partial x_2} & F_2(x_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{10}$$

$$g(t, x) = \begin{bmatrix} (1 + K_1(t)) x_1 \left(1 - x_1^2 - x_2^2\right) \\ (1 + K_2(t)) x_2 \left(1 - x_1^2 - x_2^2\right) \\ (1 + \alpha(t)) \left(F_1(x_2) + x_1 F_2(x_2) - x_3\right) \end{bmatrix}, \tag{11}$$

where $x_1 \in R$, $x_2 \in R$ and $x_3 \in R^N$. Then f is a continuously differentiable and uniformly continuous function in x, and g is a continuously differentiable and uniformly continuous function in x and t. For the system (5) with both (10) and (11), there exists a function

$$V(x) = \begin{bmatrix} V_1 \\ \vdots \\ V_m \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2^2 - 1 \\ F_1(x_2) + x_1 F_2(x_2) - x_3 \end{bmatrix} \in R^{N+1}, \tag{12}$$

where $x_1^2 + x_2^2 - 1 \in R$ and $F_1(x_2) + x_1 F_2(x_2) - x_3 \in R^N$. It is easily shown that (6) holds for (10). Now, we consider (7) with respect to (11). When $\mu = 1$,

$$\begin{aligned}V_1 \frac{\partial V_1}{\partial x} g(t, x) = -2\{(1 + K_1(t)) x_1^2 \\ + (1 + K_2(t)) x_2^2\} \left(1 - x_1^2 - x_2^2\right)^2.\end{aligned} \tag{13}$$

When $1 < \mu \le m$,

$$\begin{aligned}V_\mu \frac{\partial V_\mu}{\partial x} g(t, x) &= (1 - x_1^2 - x_2^2)(F_1(x_2) + x_1 F_2(x_2) - x_3)_{\mu-1} \\ &\cdot \Big\{ x_1(1 + K_1(t))(F_2(x_2))_{\mu-1} \\ &+ x_2(1 + K_2(t)) \left(\frac{\partial F_1}{\partial x_2} + x_1 \frac{\partial F_2}{\partial x_2}\right)_{\mu-1} \Big\} \\ &- (1 + \alpha(t))(F_1(x_2) + x_1 F_2(x_2) - x_3)_{\mu-1}^2,\end{aligned} \tag{14}$$

where $(\cdot)_i$ denotes the ith component of the vector. From Lemma 2, (13) and (14) are required to be negative and the following proposition is derived.

Proposition 1 *D_{prop} denotes a domain $D \setminus \{x \mid x_1 = 0 \wedge x_2 = 0\}$. If $K_1(t) > -1$, $K_2(t) > -1$ and $\alpha(t) > -1$, then all trajectories of (5) with (10) and (11) starting from any initial point in D_{prop} converge to a set defined as $V(x) = 0$ as $t \to \infty$.*

Example 1 *For a given limit cycle expressed by*

$$\begin{aligned}\xi(t) &= \frac{4}{3}\cos^3(t) + \sin(t)\cos(t) \\ &= F_1(\cos(t)) + \sin(t) F_2(\cos(t)),\end{aligned} \tag{15}$$

the following nonautonomous system can be synthesized:

$$\begin{aligned}\dot{x} = f(x) + g(t, x) &= \begin{bmatrix} x_2 \\ -x_1 \\ -4x_1x_2^2 - x_1^2 + x_2^2 \end{bmatrix} \\ &+ \begin{bmatrix} (1 + 0.9\sin(t)) x_1 (1 - x_1^2 - x_2^2) \\ (1 + 0.9\sin(t)) x_2 (1 - x_1^2 - x_2^2) \\ (1 + 0.9\sin(t)) (\frac{4}{3}x_2^2 + x_1x_2 - x_3) \end{bmatrix}.\end{aligned}$$

Note that $\xi(t)$ corresponds to x_3, i.e., $\lim_{t\to\infty} \parallel \xi(t) - x_3(t) \parallel = 0.$

Figure 1 shows a trajectory of Example 1 starting from $x(0) = [0, 1, -0.0286]^T$. Figure 2 depicts a transient trajectory of the system and the specified trajectory. These figures show that trajectories of the nonautonomous system which satisfies Lemma 2 converge to a specified limit cycle from any initial point.

4. Synthesis of nonautonomous systems with receding horizon control

In this section, we propose a method for determining the parameters in the nonautonomous systems (5) using receding horizon control [6]. We can obtain the optimal parameter by setting the time-depending parameters in (5) as

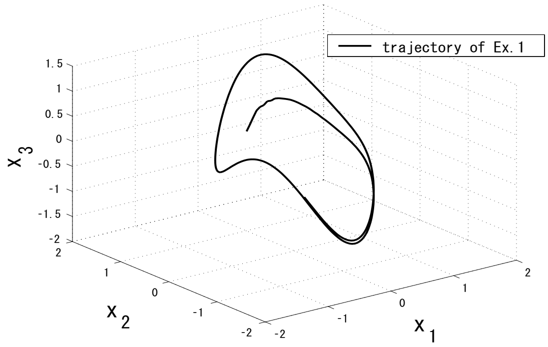

Figure 1: Trajectory of Example 1 from $x(0) = [0, 1, -0.0286]^T$.

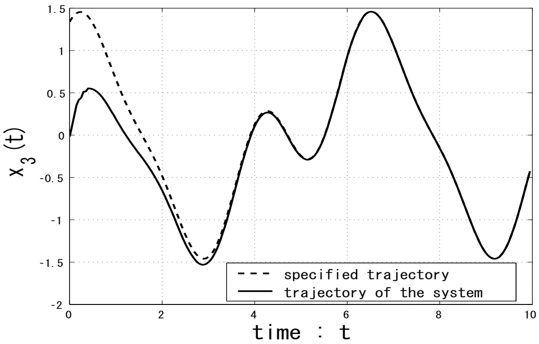

Figure 2: Transient trajectory of the system.

the control inputs and solving the receding horizon control problem.

Time-dependent parameters of $g(t, x)$ of Eq.(11) is regarded as a control input vector $u(t)$ defined as follows:

$$u(t) = \begin{bmatrix} K_1(t) \\ K_2(t) \\ \alpha(t) \end{bmatrix} \in R^{m_u}. \tag{16}$$

From Proposition 1, we have the following constraints.

$$\begin{array}{r} K_1(t) \geq K_{1\min}, \\ K_2(t) \geq K_{2\min}, \\ \alpha(t) \geq \alpha_{\min}, \end{array} \tag{17}$$

where $K_{1\min}$, $K_{2\min}$, and $\alpha_{\min}$ are constants greater than -1. Consider the following differential equations:

$$\left\{ \begin{array}{l} \dot{x} = f(x(t)) + g(u(t), x(t)), \\ \dot{p} = f(p(t)), \end{array} \right. \tag{18}$$

where $x(t) \in R^n$ is a state vector at time t and $p(t) \in R^n$ represents a reference point vector on the limit cycle at time t. We define an augmented vector as follows:

$$X(t) = \begin{pmatrix} x(t) \\ p(t) \end{pmatrix} \in R^{2n}. \tag{19}$$

From (19), (18) is rewritten as the following differential equation:

$$\dot{X} = \begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} f(x) + g(u, x) \\ f(p) \end{pmatrix} = F(u(t), X(t)). \tag{20}$$

For (20), the following index function with a receding horizon $T > 0$ is introduced.

$$J := \varphi(t + T) + \int_t^{t+T} L(u(\tau), X(\tau)) d\tau, \tag{21}$$

$$\begin{aligned} \varphi(t + T) := & (Q_1 X(t + T) - Q_2 X(t + T))^T \\ & \cdot (Q_1 X(t + T) - Q_2 X(t + T)), \end{aligned} \tag{22}$$

$$\begin{aligned} L(u(t), X(t)) := & (Q_1 X(\tau) - Q_2 X(\tau))^T \\ & \cdot (Q_1 X(\tau) - Q_2 X(\tau)) + u^T u, \end{aligned} \tag{23}$$

where $Q_1 = [I|\mathbf{0}]$, $Q_2 = [\mathbf{0}|I]$, and I and **0** are the $n \times n$ identity and zero matrix, respectively. In (21), the first term is the distance between the state vector and the reference point vector at time $t + T$. The second term represents the sum of the square measure of the state, the reference point, and the control inputs from time t to $t + T$.

We determine the optimal control input $u_{opt}(t)$ of the nonautonomous system (20) so as to minimize the index function (21) at time t.

From the viewpoint of such a computational complexity, an efficient numerical method for receding horizon control has been proposed [6]. In this method, the optimal control problem is discretized; the horizon T is divided into N steps and the optimal control input of each sampling time is characterized by the following equations:

$$X_{i+1}^*(t) = X_i^*(t) + F(u_i^*(t), X_i^*(t)) \Delta\tau, \tag{24}$$

$$X_0^*(t) = X(t), \tag{25}$$

$$H_u(X_i^*(t), \lambda_{i+1}^*(t), u_i^*(t)) = 0, \tag{26}$$

$$\lambda_i^*(t) = \lambda_{i+1}^*(t) + {H_X}^T(X_i^*(t), \lambda_{i+1}^*, u_i^*(t)) \Delta\tau, \tag{27}$$

$$\lambda_N^*(t) = {\varphi_X}^T(X_N^*(t)), \tag{28}$$

where $\Delta\tau := T/N$, $X_i^*(t) \in R^{2n}$ expresses the state of the ith step for a discrete optimal control problem starting with $X(t)$, and $\lambda_i^*(t) \in R^{2n}$ and $u_i^*(t) \in R^{m_u}$ represents the costate and the control inputs of the ith step, respectively. Let $H(X, \lambda, u)$ be a Hamiltonian defined by

$$H(X, \lambda, u) := L(u, X) + \lambda^T F(u, X). \tag{29}$$

Let a vector of the series of the control input $U(t)$ and the map $P_0(U(t))$ be as follows:

$$U(t) := \left[\begin{array}{cccc} u_0^{*T} & u_1^{*T} & \cdots & {u_{N-1}^*}^T \end{array} \right]^T \in R^{m_u N}, \tag{30}$$

$$P_0(U(t)) := u_0^*(t). \tag{31}$$

If $U(t)$ and $X(t)$ are given, we can determine $\{X_i^*(t)\}_{i=0}^N$ and $\{\lambda_i^*(t)\}_{i=0}^N$ from (24), (25), (27), and (28). Therefore, (26) can be reduced to the following equation:

$$\begin{aligned} & Y(U(t), X(t), t) \\ & := \begin{bmatrix} H_u^T(X_0^*(t), \lambda_1^*(t), u_0^*(t)) \\ \vdots \\ H_u^T(X_{N-1}^*(t), \lambda_N^*(t), u_{N-1}^*(t)) \end{bmatrix} = 0. \end{aligned} \tag{32}$$

Solving (32) for $X(t)$ sampled at each time, we can obtain $U(t)$ and determine the optimal control input $u_{opt}(t) = P_0(U(t))$ at time t.

Example 2 *Consider the following nonautonomous system*

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} f(x(t)) + g(u(t), x(t)) \\ f(p(t)) \end{bmatrix},$$
$$= \begin{bmatrix} x_2 + (1+K_1)x_1(1 - x_1^2 - x_2^2) \\ -x_1 + (1+K_2)x_2(1 - x_1^2 - x_2^2) \\ -4x_1x_2^2 - x_1^2 + x_2^2 + (1+\alpha)(\frac{4}{3}x_2^3 + x_1x_2 - x_3) \\ p_2 \\ -p_1 \\ -4p_1p_2^2 - p_1^2 + p_2^2 \end{bmatrix}. \quad (33)$$

Note that $x_3(t)$ converges to the same limit cycle (15) as Example 1. We adopt receding horizon control to (33) with the following index function:

$$J := \varphi(t+T) + \int_t^{t+T} L(X(\tau), u(\tau))d\tau,$$
$$\varphi(t+T) := (x_1(t+T) - p_1(t+T))^2 + (x_2(t+T) - p_2(t+T))^2 + (x_3(t+T) - p_3(t+T))^2,$$
$$L(X(t), u(t)) := (x_1(t) - p_1(t))^2 + (x_2(t) - p_2(t))^2 + (x_3(t) - p_3(t))^2 + K_1(t)^2 + K_2(t)^2 + \alpha(t)^2.$$

Figure 3 shows the transient trajectory in the 3-dimensional space starting from $X(0) = [0, 1, -0.0286, 0, 1, 1.3333]^T$. Figure 4 illustrates the distance between the optimal trajectory and the uncontrolled trajectory with $u(t) \equiv 0$, respectively. Figure 5 shows the optimal input $\alpha(t)$ of receding horizon control, i.e., changes of the time-depending parameter $\alpha(t)$ in $g(u(t), x(t))$. Figures 4 and 5 show that the controlled trajectory by receding horizon control converges to the specified limit cycle faster than the uncontrolled trajectory.

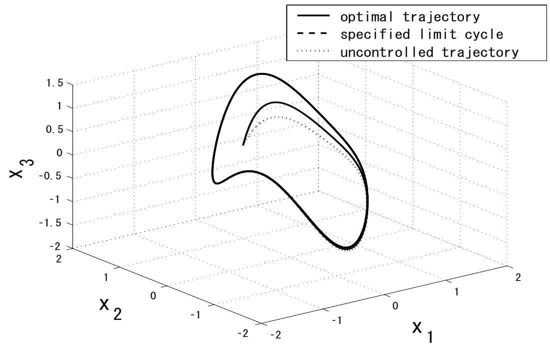

Figure 3: Trajectories of Example 2 in the state space.

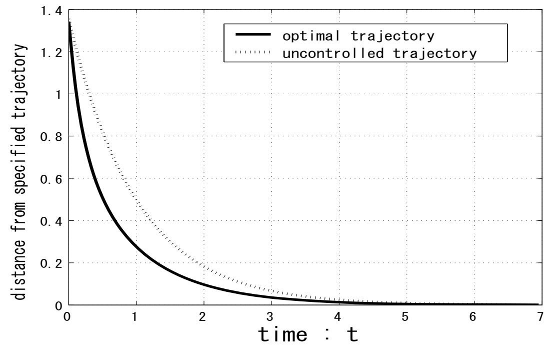

Figure 4: Distance from specified trajectory.

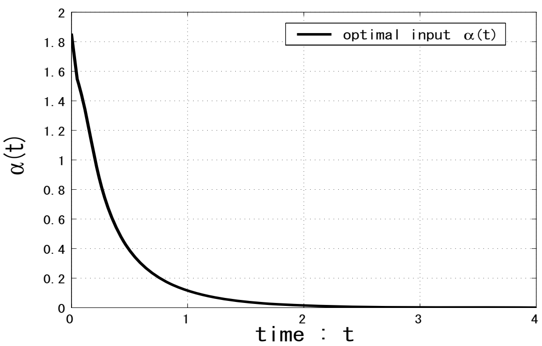

Figure 5: Optimal inputs $\alpha(t)$ of Example 2

5. Conclusions

In this paper, we proposed a synthesis method of nonautonomous systems $\dot{x} = f(x) + g(t, x)$ with a stable limit cycle. As an extension of Green' method, we showed that transient trajectories of the system starting from any initial point converge to the specified limit cycle. Moreover, we proposed a method for determining time-dependent parameters in $g(t, x)$ by receding horizon control. We also considered 3-dimensional nonautonomous system as an example and showed the efficiency of the method.

It is future work to apply the method to walking pattern generations of humanoid robots so as to adapt the change of their environment.

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