Lyapunov Bundle of Saddle Quasi-Periodic Solutions

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Abstract—Dynamical systems, such as coupled oscillators, which produce quasi-periodic solutions are ubiquitous, and there exists many complex related bifurcations. These bifurcations of quasi-periodic solutions attract much attention in recent years. However, bifurcation analysis of saddle quasi-periodic solutions is difficult and not so much discussed. In this paper, a calculation method for the Lyapunov bundle of saddle quasi-periodic solutions is introduced.

1. Introduction

Newton's method for saddle quasi-periodic solutions was developed [1], [2]. This method uses frequency characteristics of the solution and can be applied to higher-dimensional saddle quasi-periodic solutions. By using this method, we succeeded in calculating the Lyapunov bundle of a saddle quasi-periodic solution in a discrete-time dynamical system. The Lyapunov bundle was developed for analyzing quasi-periodic bifurcations [3]. It is a set of Lyapunov vectors on a solution and can classify local bifurcation types from its topology. In this paper, we demonstrate the Lyapunov bundle of saddle quasi-periodic solutions.

2. System Equation

As demonstration, we use three circulant coupled map $\mathbb{R}^3 \to \mathbb{R}^3$ defined as follows:

$$\begin{pmatrix} x_0(t+1) \\ x_1(t+1) \\ x_2(t+1) \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & c_3 \\ c_3 & c_1 & c_2 \\ c_2 & c_3 & c_1 \end{pmatrix} \begin{pmatrix} g(x_0(t)) \\ g(x_1(t)) \\ g(x_2(t)) \end{pmatrix}, \qquad (1)$$

where

$$\begin{aligned} g(x) &= -x(ax-1+a), a = 1-\mu, \\ e &= r\cos(\theta\pi/180)/\mu, \\ d &= r\sin(\theta\pi/180)/\mu, \\ c_1 &= (1+2e)/3, \\ c_2 &= (1-e-\sqrt{3}d)/3, \\ c_3 &= (1-e+\sqrt{3}d)/3. \end{aligned} \qquad (2)$$

This system has three parameters: r, μ, and θ. We choose r as the bifurcation parameter. The other parameters are fixed at $\mu = -0.5$ and $\theta = 46$. This system shows a double covering (torus loop doubling) bifurcation of a 1-dimensional quasi-periodic solution (invariant closed curve). Figure 1 represents the one-parameter Lyapunov diagram in terms of r. Before the double covering bifurcation, there is a stable one-loop quasi-periodic solution as shown in Fig. 2.

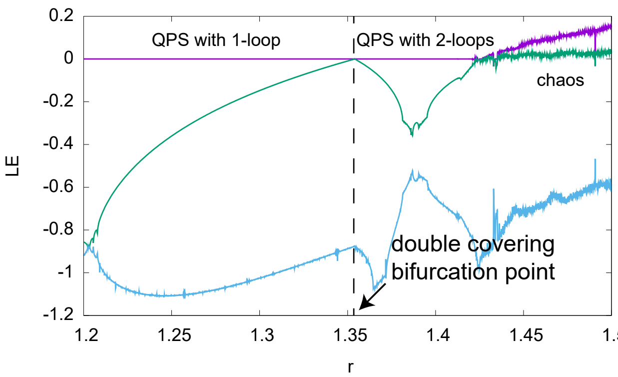

Figure 1: The one-parameter Lyapunov diagram of the double covering bifurcation of the quasi-periodic solution (QPS) in terms of r.

After the bifurcation, the solution doubles its number of loops as shown in Fig. 3. For visibility, we project $\mathbf{x}(= (x_0, x_1, x_2))$ onto $\mathbf{x}'(= (x'_0, x'_1, x'_2))$ via a projection matrix Eq. (3) for all the plots.

$$\begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \cos(0/3) & \cos(2\pi/3) & \cos(4\pi/3) \\ \sin(0/3) & \sin(2\pi/3) & \sin(4\pi/3) \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}. \qquad (3)$$

3. Results

In our expectation, there may exist a saddle quasi-periodic solution surrounded by the stable two-looped quasi-periodic solution after the bifurcation. Then, we applied continuation for the solution from stable to unstable parameter values by using Newton's method for quasi-periodic solutions [1], [2] and our Lyapunov bundle calculation method [3].

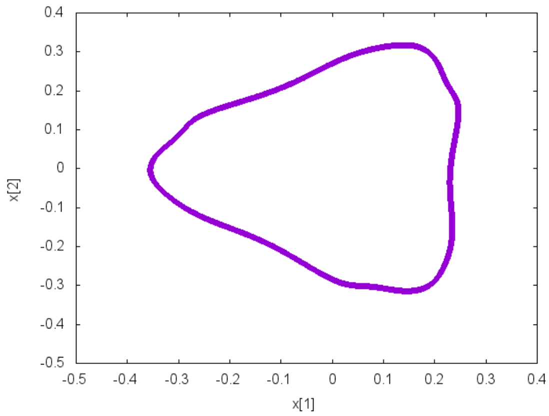

Figure 2: Stable one-looped quasi-periodic solution before the double covering bifurcation at $r = 1.3$.

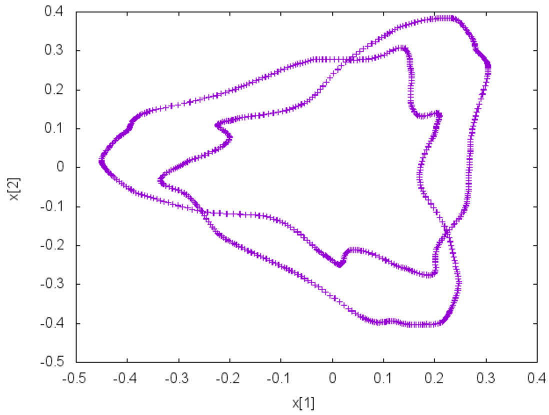

Figure 3: Stable two-looped quasi-periodic solution after the double covering bifurcation at $r = 1.4$.

This Newton's method is one of the parameterization methods for quasi-periodic solutions. The procedure of the method is as follows: First, convert the original phase space to a 1-dimensional phase space parameterized by the angle θ (mod 2π) from the center of the solution. Next, calculate Fourier coefficients of the solution in the converted phase space. Finally, We can apply Newton's method to these coefficients and also apply continuation to the solution.

As a result, the expected saddle solution was successfully obtained with its Lyapunov bundle for unstable direction as shown in Fig. 4. From this figure, we can observe that the saddle is surrounded by the stable two-looped solution and the Lyapunov bundle for the unstable direction is facing to the stable solution. The bundle has Möbius bundle shape, because the bifurcation is double covering. Furthermore, the Lyapunov bundle direction suggests that the unstable manifold of the saddle connects to the stable solution.

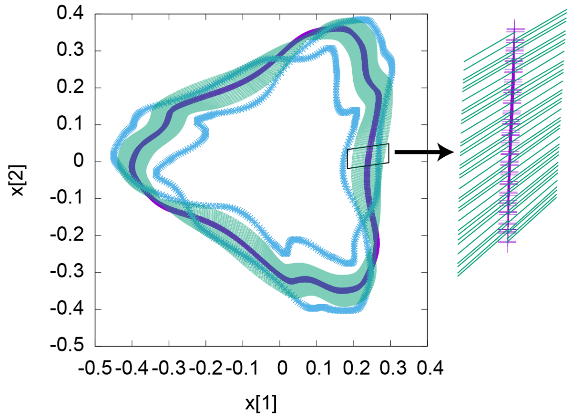

Figure 4: Saddle one-looped quasi-periodic solution (purple) and its Lyapunov bundle for the unstable direction (green) and stable two-looped quasi-periodic solution (light blue) after the double covering bifurcation at $r = 1.4$. Enlarged part of plot is shown in right.

4. Conclusion

We have combined Newton's method for saddle quasi-periodic solution and our Lyapunov bundle method. Then, we succeeded in obtaining the Lyapunov bundle of the saddle. Our future works are the calculation of unstable manifolds of the saddle based on the Lyapunov bundle and the analyses of global bifurcations of higher-dimensional quasi-periodic solutions.

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