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Parametric Stability

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Abstract—The objective of this paper is to consider parametric stability of dynamical systems. Conditions to guarantee that system preserves a stability region despite the shift of the equilibrium location caused by changes of uncertainty parameters and constant reference inputs are shown.

1. Introduction

For almost all dynamical models, the existence of equilibria and their stability are the two basic problems of analysis. The standard approach is first to locate the equilibria, then select one that is interest, translate it to the origin, and lastly determine its stability properties. However, this approach break down when parametric uncertainties are present because of modeling inaccuracies or changes in the environment of the model. Each time a parameter is changed the original equilibrium may shift to a new location or disappear altogether, thus making the stability analysis of the translated equilibrium at the origin either imprecise or entirely useless. For this reason, the concept of parametric stability was introduced [1].

In the paper, a region $\mathcal{P}$ of uncertain parameters and constant reference inputs is assigned, and parametric stability is considered. Since asymptotic stability in the large is easy to require some restrictive conditions [1]-[5], we derive a stability region $\mathcal{X}_0$ of initial state for a given region $\mathcal{P}$ of uncertain parameters and constant reference.

2. Parametric Stability

Let us consider a nonlinear time-invariant system

$$\mathcal{S}: \ \dot{x} = f(x, p), \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state of $\mathcal{S}$ at time $t \in \mathbf{R}$, $p \in \mathbf{R}^m$ is a constant parameter vector, and $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ is a sufficiently smooth function so that for any $p \in \mathbf{R}^m$ and initial state $x_0 \in \mathbf{R}^n$ at time $t_0 = 0$, Eq. (1) has the unique solution $x(t; x_0, p)$. We also assume that for some nominal value p^* of the parameter vector p, there is an equilibrium state x^*, that is,

$$f(x^*, p^*) = 0 \tag{2}$$

and x^* is stable. Suppose that the parameter vector p is changed from p^* to another value. The question arises: Does there exist a new equilibrium x^e of Eq. (1) and how far is it from x^*? If x^e exists, is it stable as x^* was, or is stability destroyed by the change of p? To provide answers to these questions, we consider the equilibrium $x^e : \mathbf{R}^m \to \mathbf{R}^n$ as a function $x^e(p)$.

In this paper, we consider parametric stability in the following sense:

Definition 1 A system $\mathcal{S}$ is said to be parametrically asymptotically stable with respect to a region $\mathcal{X}_0 \times \mathcal{P} \subset \mathbf{R}^n \times \mathbf{R}^m$ if for any $p \in \mathcal{P}$,

(i) there exists a unique equilibrium $x^e(p) \in \mathcal{X}_0$;

(ii) for any $x_0 \in \mathcal{X}_0$, $x(t; x_0, p) \in \mathcal{X}_0$ for all $t \geqq 0$;

(iii) the equilibrium $x^e(p)$ is stable; and

(iv) for any $x_0 \in \mathcal{X}_0$, $\lim_{t\to\infty} x(t; x_0, p) = x^e(p)$.

3. Analysis

Concerning the requirement (i) of Definition 1, using the degree theory (see Appendx A.1), the following results were shown in [1].

Lemma 1 Assume that there exists an open and bounded set C containing x^*, and a bounded and simply connected set $\mathcal{P}$ containing p^*, where $x^* \in \mathbf{R}^n$ and $p^* \in \mathbf{R}^m$ are vectors satisfying Eq. (2). Then, for each $p \in \mathcal{P}$, the equation

$$f(x, p) = 0 \tag{3}$$

has a solution $x^e(p) \in C$ if:

(i) x^* is a unique solution of Eq. (2) in C and $\det D_x f(x^*, p^*) \neq 0$, where $D_x f(x, p)$ denotes the partial derivative of f with respect to x; and

(ii) $f(x, p') \neq 0$ for all $x \in \partial C$ and $p' \in \mathcal{P}$, where ∂C denotes the boundary of C.

Let $|\cdot|$ be a given norm in $\mathbf{R}^n$ and $|\cdot|'$ a given norm in $\mathbf{R}^m$. Define two closed balls by

$$\overline{\mathcal{B}}(x^*; \hat{r}) = \{x \in \mathbf{R}^n : \ |x - x^*| \leqq \hat{r}, \ \ \hat{r} > 0\}, \tag{4}$$

$$\overline{\mathcal{B}}(p^*; q^*) = \{p \in \mathbf{R}^m : \ |p - p^*|' \leqq q^*, \ \ q^* > 0\}. \tag{5}$$

Corollary 1 Let $\mathcal{X}_0 = \overline{\mathcal{B}}(x^*; \hat{r})$ and $\mathcal{P} = \overline{\mathcal{B}}(p^*; q^*)$.
Assume the following:

$$\det D_x f(x^*.p^*) \neq 0, \tag{6}$$
$$\exists\, \hat{K} > 0 : \quad |f(x^*, p') - f(x^*.p^*)| \leqq \hat{K} \quad \forall\, p' \in \mathcal{P}, \tag{7}$$
$$\exists\, \alpha > 0 : \quad |f(x, p') - f(x^*, p')| \geqq \alpha|x - x^*| \quad \forall\, x \in \mathcal{X}_0, \quad \forall\, p' \in \mathcal{P}, \tag{8}$$
$$\alpha\hat{r} > \hat{K}. \tag{9}$$

Let ε be a small positive number such that $\alpha\hat{r} > \hat{K} + \varepsilon$, and let $r^* \in (0, \hat{r})$ be a number such that $\alpha r^* = \hat{K} + \varepsilon$. Then, for any $p \in \mathcal{P} = \overline{\mathcal{B}}(p^*; q^*)$, Eq. (3) has a solution $x^e(p) \in C = \mathcal{B}(x^*; r^*) = \{x \in \mathbf{R}^n : |x - x^*| < r^*\}$.

Let us introduce the λ functional, which is defined by

$$\lambda(f(\cdot, p); x, y) = \lim_{\Delta\downarrow 0} \frac{|x + \Delta f(x, p) - [y + \Delta f(y, p)]| - |x - y|}{\Delta|x - y|}. \tag{10}$$

When $y = x^e(p)$, $\lambda(f(\cdot, p); x, x^e(p))|x - x^e(p)|$ is the upper right-hand derivative of $V(x) = |x - x^e(p)|$ along the solution of (1): $\dot{x} = f(x, p) - f(x^e(p), p) = f(x, p)$. Therefore, the stability parts (*ii*) and (*iii*) of Definition 1 will be established by using the λ functional. We have the following:

Theorem 1 Let $\mathcal{X}_0 = \overline{\mathcal{B}}(x^*; \hat{r})$ and $\mathcal{P} = \overline{\mathcal{B}}(p^*; q^*)$.
In addition to conditions (6) and (7), we assume the following:

$$\exists\, \alpha > 0 : \quad \alpha\hat{r} > \hat{K}, \quad \lambda(f(\cdot, p); x, y) \leqq -\alpha \quad \forall\, x, y \in \mathcal{X}_0, \quad x \neq y, \quad \forall\, p \in \mathcal{P}. \tag{11}$$

Then, $\mathcal{S}$ is parametrically asymptotically stable with respect to $\mathcal{X}_0 \times \mathcal{P}$. Moreover, for any $p \in \mathcal{P}$, the equilibrium $x^e(p)$ exists in $C = \mathcal{B}(x^*; r^*) \subseteq \mathcal{X}_0$, where $r^* \in (0, \hat{r})$ is the number such that $\alpha r^* = \hat{K} + \varepsilon$ and ε is a small positive number such that $\alpha\hat{r} > \hat{K} + \varepsilon$,

The condition (11) of Theorem 1 can be replaced by less conservative condition:

$$\exists\, \alpha > 0 : \quad \alpha\hat{r} > \hat{K} \quad \lambda(f(\cdot, p); x, y) \leqq -\alpha \quad \forall\, x \in \mathcal{X}_0, \quad \forall\, y \in C, \quad x \neq y, \quad \forall\, p \in \mathcal{P}. \tag{12}$$

To examine conditions (11) or (12), we need to scan 2 vectors x and y in $\mathcal{X}_0$. We may have a (sufficient) condition which can be examined more easily.

Theorem 2 Let $\mathcal{X}$ be a bounded convex set including x^*, and $\mathcal{P} = \overline{\mathcal{B}}(p^*; q^*)$. In addition to the condition (7), we assume the following:

$$\exists\, \{A_q \in \mathbf{R}^{n\times n}\}_{q=1}^{Q} : \; D_x(x, p) \in \operatorname{conv}\{A_q\}_{q=1}^{Q} \quad \forall\, x \in \mathcal{X}, \; \forall\, p \in \mathcal{P} \tag{13}$$
$$\exists\, \alpha > 0 : \quad \lambda(F_q(\cdot); \xi, 0) \leqq -\alpha \quad \forall\, \xi \in \mathbf{R}^n, \quad \forall\, q \in \overline{Q} = \{1, 2, \cdots, Q\}, \tag{14}$$

where $F_q(\xi) = A_q\xi$.

Let

$$\Omega(r; x^*) = \{x \in \mathbf{R}^n : |x - x^*| \leqq r\}, \tag{15}$$
$$\hat{r} = \max\{r > 0 : \Omega(r; x^*) \subseteq \mathcal{X}\}. \tag{16}$$

If the condition (9) is met, the conclusions of Theorem 1 hold.

A candidate of a norm $|\cdot|$ used to define λ-functional is $|x| = \sqrt{x^T P x} = |P^{1/2}x|_2$, where P is a solution of the following LMI.

$$PA_q + A_q^T P + 2\alpha I \prec 0, \quad P = P^T \succ 0, \quad q \in \overline{Q}. \tag{17}$$

In the above equation, $A \prec 0$ and $P \succ 0$ mean that A is negative definite and P is positive definite, respectively.

Another candidate of a norm $|\cdot|$ is given by

$$|x| = \max\{|h_\ell^T x|, \; \ell = 1, 2, \ldots, L\}, \tag{18}$$

where h_ℓ is the normal vector of a polytope $\mathcal{X}_0$ including 0 as an interior point and defined by

$$\mathcal{X}_0 = \{x : \; h_\ell^T x \leqq 1, \; \ell = 1, 2, \ldots, L\}. \tag{19}$$

An algorithm constructing $\mathcal{X}_0$ satisfying (14) is given in [8].

4. Concluding Remark

In this paper, we consider parametrically asymptotically stability with respect to a region $\mathcal{X}_0 \times \mathcal{P}$ of nonlinear systems. This requires that $\mathcal{X}_0$ must be a stability region of the system under changes of uncertainty parameters and constant reference inputs.

We show two candidates of of a norm $|x|$. Note that the norm $|x|$ depends on the choice α in (17) or (14).

Appendix

A.1 The degree of a mapping [6]

Throughout this section, let $\mathcal{D} \subseteq \mathbf{R}^n$ be an open set, C an open, bounded set with $\overline{C} \subset \mathcal{D}$, $F : \mathcal{D} \subseteq \mathbf{R}^n \to \mathbf{R}^n$, $y \notin F(\partial C)$ a given point. and

$$\hat{\sigma} = \min\{|F(x) - y|_2 : \; x \in \partial C\}. \tag{20}$$

For given $\alpha > 0$, let W_α be the set of all real functions $\varphi : [0, \infty) \subset \mathbf{R}^1 \to \mathbf{R}^1$ which are continuous on $[0, \infty)$ and for which there exists a $\delta \in (0, \alpha)$ such that $\varphi(t) = 0$ whenever $t \notin [\delta, \alpha]$. We call every $\varphi \in W_\alpha$ a weight function of index α. Define

$$W_\alpha^1 = \left\{\varphi \in W_\alpha : \; \int_{\mathbf{R}^n} \varphi(|x|_2)dx = 1\right\}.$$

When F is continuously differentiable on the open set $\mathcal{D}$, for any weight function φ of index $\alpha < \hat{\sigma}$, define the mapping $\phi : \mathbf{R}^n \to \mathbf{R}^1$ by

$$\phi(x) = \begin{cases} \varphi(|F(x) - y|_2) \det F'(x), & x \in C \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

Then the integral $d_\varphi(F,C,y) = \int_{\mathbf{R}^n} \phi(x)dx$ is called the degree integral of F on C with respect to y and the weight function φ.

Lemma 2 [6] Let F be continuously differentiable on the open set $\mathcal{D}$, and $\Gamma = \{x \in C : F(x) = y\}$. Suppose that $F'(x)$ is nonsingular for all $x \in \Gamma$. Then, Γ consists of at most finitely many points, and there exists an $\hat{\alpha}$ with $0 < \hat{\alpha} \leqq \hat{\sigma}$ such that, for any $\varphi \in W^1_\alpha$ with $\alpha \in (0, \hat{\alpha})$,

$$d_\varphi(F,C,y) = \begin{cases} \sum_{j=1}^{m} \text{sgn}\det F'(x^j), & \text{if } \Gamma = \{x^1, \ldots, x^m\} \\ 0, & \text{if } \Gamma \text{ is empty.} \end{cases}$$

It can be shown that $d_{\varphi_1}(F,C,y) = d_{\varphi_2}(F,C,y)$ for all $\varphi_1, \varphi_2 \in W_\alpha$ when $\alpha < \hat{\sigma}$. The degree of F at any point $y \notin F(\partial C)$ withe respect to C is defined by

$$\deg(F,C,y) = d_\varphi(F,C,y), \tag{22}$$

where $\varphi \in W^1_\alpha$ and $\alpha \in (0, \hat{\sigma})$.

When F is continuous on $\overline{C}$, the degree of F at y with respect to C is defined by

$$\deg(F,C,y) = \lim_{k\to\infty} \deg(F_k,C,y), \tag{23}$$

where $F_k : \mathcal{D} \subseteq \mathbf{R}^n \to \mathbf{R}^n$ is any sequence of maps which are continuously differentiable on $\mathcal{D}$ and for which $\lim_{k\to\infty} |F_k - F|_\infty = 0$.

Lemma 3 [6] (Kronecker Theorem) Let F is continuous on $\overline{C}$. If $\deg(F,C,y) \neq 0$, then the equation $F(x) = y$ has a solution in C.

Lemma 4 [6] (Homotopy Invariance Theorem) Let $H : \overline{C} \times [0,1] \subseteq \mathbf{R}^{n+1} \to \mathbf{R}^n$ be continuous. Suppose that $z \in \mathbf{R}^n$ satisfies $H(x,t) \neq z$ for all $(x,t) \in \partial C \times [0,1]$. Then, $\deg(H(\cdot,t),C,z)$ is constant for $t \in [0,1]$.

A.2 Proof of Corollary 1

By the condition (8), it is obvious that x^* is a only one solution of the equation $f(x,p^*) = 0$ in C. Therefore, conditions (8) and (6) mean that the condition (*i*) of Lemma 1 is met.

Given arbitrary $p \in \mathcal{P}$. Define $H : \overline{C} \times [0,1] \to \mathbf{R}^n$ by

$$H(x,t) = f(x,(1-t)p^* + tp). \tag{24}$$

Note that $[(1-t)p^* + tp] \in \mathcal{P}$ for all $t \in [0,1]$ and $p \in \mathcal{P}$, and that H is continuous and $H(x^*,0) = f(x^*,p^*) = 0$.

Observe that

$$\begin{aligned} |H(x,t)| &= |H(x,t) - H(x^*,0)| \\ &= |f(x,(1-t)p^* + tp) - f(x^*,(1-t)p^* + tp) \\ &\quad + f(x^*,(1-t)p^* + tp) - f(x^*,p^*)| \\ &\geqq |f(x,(1-t)p^* + tp) - f(x^*,(1-t)p^* + tp)| \\ &\quad - |f(x^*,(1-t)p^* + tp) - f(x^*,p^*)|. \end{aligned}$$

Then, by conditions (7) - (9), we have

$$\begin{aligned} |H(x,t)| &\geqq \alpha|x - x^*| - \hat{K} \geqq \alpha r^* - \hat{K} \\ &= \varepsilon > 0 \ \ \forall\, x \in \partial C, \ \ \forall\, p \in \mathcal{P}, \ \ \forall\, t \in [0,1], \end{aligned}$$

and, hence, we have $H(x,1) = f(x,p) \neq 0$ for all $x \in \partial C$ and $p \in \mathcal{P}$, that is, the condition (*ii*) of Lemma 1 is met. We have the result by Lemma 1. ■

A.3 The λ functional [7]

Let $F : \mathbf{R}^n \to \mathbf{R}^n$. Define the λ functional at $x,y \in \mathbf{R}^n$, $x \neq y$ by

$$\lambda(F;x,y) = \lim_{\Delta\downarrow 0} \frac{\gamma(I + \Delta F;x,y) - 1}{\Delta}, \tag{25}$$

$$\gamma(I + \Delta F;x,y) = \frac{|(I + \Delta F)(x) - (I + \Delta F)(y)|}{|x - y|}. \tag{26}$$

Lemma 5 [7] Let $F,G : \mathbf{R}^n \to \mathbf{R}^n$. The λ functional has the following properties for all $x,y \in \mathbf{R}^n$, $x \neq y$.

(*i*) $\lambda(I;x,y) = 1$, $\lambda(-I;x,y) = -1$;

(*ii*) $-\gamma(F;x,y) \leqq -\lambda(-F;x,y) \leqq \lambda(F;x,y) \leqq \gamma(F;x,y)$;

(*iii*) $\lambda(\beta F;x,y) = \beta\lambda(F;x,y) \quad \forall\beta \geqq 0$;

(*iv*) $\lambda(F + G;x,y) \leqq \lambda(F;x,y) + \lambda(G;x,y)$;

(*v*) $|F(x) - F(y)| \geqq \max\{\lambda(F;x,y), \lambda(-F;x,y)\}\,|x - y|$.

A.4 Proof of Theorem 1

From the property (*ii*) in Lemma 5 and the condition (11) of Theorem 1, we have

$$-\lambda(-f(\cdot,p);x,y) \leqq \lambda(f(\cdot,p);x,y) \leqq -\alpha \ \ \forall\, x,y \in \mathcal{X}_0, x \neq y.$$

Then, by applying the property (*v*) in Lemma 5, we obtain

$$|f(x,p) - f(y,p)| \geqq \lambda(-f(\cdot,p);x,y)|x - y| \geqq \alpha|x - y| \quad \forall\, x,y \in \mathcal{X}_0. \tag{27}$$

Thus, all the conditions of Corollary 1 are satisfied, and, hence, the condition (*i*) of Definition 1 holds.

Given arbitrary $p \in \mathcal{P}$ and let $x(t)$ be a solution of

$$\dot{x}(t) = f(x(t),p), \ \ x(0) = x_0 \in \mathcal{X}_0. \tag{28}$$

We will show the condition (*ii*) of Definition 1. Suppose that $x(t) \in \operatorname{int}\mathcal{X}_0$ for $t \in [0,\hat{t})$. If $\hat{t} = \infty$, then $x(t) \in \mathcal{X}_0$ for all $t \geqq 0$. Suppose now $\hat{t} < \infty$ and $x(\hat{t}) \in \partial\mathcal{X}_0$. Let $e(t) = x(t) - x^*$, $V(x) = |e|$, $\tilde{V}(t) = V[e(t)] = |e(t)|$. Then, by the conditions (11) and (7), we have

$$\dot{e}(t) = \dot{x}(t) = f(x(t),p) - f(x^*,p^*) \tag{29}$$

$$\begin{aligned} &\lim_{\Delta\downarrow 0} \frac{\tilde{V}(\hat{t} + \Delta) - \tilde{V}(\hat{t})}{\Delta} = \lim_{\Delta\downarrow 0} \frac{|e(\hat{t} + \Delta)| - |e(\hat{t})|}{\Delta} \\ &\leqq \lim_{\Delta\downarrow 0} \sup_{\Delta>0} \frac{|e(\hat{t}) + \Delta[f(x(\hat{t}),p) - f(x^*,p^*)]| - |e(\hat{t})|}{\Delta} \\ &\leqq \lim_{\Delta\downarrow 0} \frac{|x(\hat{t}) - x^* + \Delta[f(x(\hat{t}),p) - f(x^*,p)]| - |x(\hat{t}) - x^*|}{\Delta} \\ &\qquad + |f(x^*,p) - f(x^*,p^*)| \\ &\leqq \lambda(f(\cdot,p);x(\hat{t}),x^*)|x(\hat{x}) - x^*| + \hat{K} \leqq -\alpha|x(\hat{t}) - x^*| + \hat{K} \\ &= -\alpha\hat{r} + \hat{K} < 0, \end{aligned} \tag{30}$$

Therefore, there exists a small positive number $\hat{\tau}$ such that $\tilde{V}(\hat{t}+\tau) = |x(\hat{t}+\tau) - x^*| \leqq \hat{r}$ for all $\tau \in [0, \hat{\tau})$. Repeating this, we can conclude that the condition (*ii*) of Definition 1 holds.

Finally, we will consider the conditions (*iii*) and (*iv*) of Definition 1. Let $\tilde{e}(t) = x(t) - x^e$, where $x^e = x^e(p)$ is the equilibrium of the system (1), that is, it is a solution of (3). Then, we have

$$\dot{\tilde{e}}(t) = \dot{x}(t) = f(x(t)) - f(x^e, p) \qquad (31)$$

Let $W(\tilde{e}) = |\tilde{e}| = |x - x^e|$, $\tilde{W}(t) = W(\tilde{e}(t))$. Then, by the condition (11), we have

$$\begin{aligned}
&\lim_{\Delta\downarrow 0} \frac{\tilde{W}(t+\Delta) - \tilde{W}(t)}{\Delta} = \lim_{\Delta\downarrow 0} \frac{|\tilde{e}(t+\Delta)| - |\tilde{e}(t)|}{\Delta} \\
&\leqq \lim_{\Delta\downarrow 0} \sup_{\Delta>0} \frac{|\tilde{e}(t) + \Delta[f(x(t),p) - f(x^e,p)]| - |\tilde{e}(t)|}{\Delta} \\
&\leqq \lim_{\Delta\downarrow 0} \frac{|x(t) - x^e + \Delta[f(x(t),p) - f(x^e,p)]| - |x(t) - x^e|}{\Delta} \\
&= \lambda(f(\cdot,p); x(t), x^e)|x(\hat{x}) - x^e| \leqq -\alpha|x(t) - x^e| = -\alpha\tilde{W}(t).
\end{aligned}$$

Since the condition (*ii*) of Definition 1 holds, the above inequality implies that the conditions (*iii*) and (*iv*) of Definition 1 are hold. ■

A.5 Proof of Theorem 2

1) We will show that (6) is met.

Given arbitrary $x \in \mathcal{X}$ and $p \in \mathcal{P}$. By the condition (13), there exist numbers $\{\beta_q(x,p) \in [0,1]\}_{q=1}^{Q}$ such that

$$D_x f(x,p) = \sum_{q=1}^{Q} \beta_q(x,p) A_q, \quad \sum_{q=1}^{Q} \beta_q(x,p) = 1. \qquad (32)$$

Let $F(\xi; x, p) = D_x f(x,p)\xi$ and $F_q(\xi) = A_q\xi$. By the condition (14), (32), properties (*iii*) and (*iv*) in Lemma 5, we have

$$\begin{aligned}
&\lambda(F(\cdot; x,p); \xi, 0) = \lambda(\sum_{q=1}^{n} \beta_q(x,p) F_q; \xi, 0) \\
&\leqq \sum_{q=1}^{n} \beta_q(x,p) \lambda(F_q; \xi, 0) \leqq \sum_{q=1}^{n} \beta_q(x,p)(-\alpha) = -\alpha. \qquad (33)
\end{aligned}$$

Given arbitrary matrix $A \in \mathbf{R}^{n\times n}$ and let $\tilde{F}(\xi) = A\xi$, $\lambda_i(A) = (a_i + jb_i) \in \mathbf{C}$ be the i-th eigenvalue of A, where $j = \sqrt{-1}$, and $\xi_i \neq 0$ be an eigen vector corresponding to $(a_i + jb_i)$, that is, $A\xi_i = (a_i + jb_i)\xi_i$. Then, we have

$$\begin{aligned}
\lambda(\tilde{F}; \xi_i, 0) &= \lim_{\Delta\downarrow 0} \frac{|\xi_i + \Delta A\xi_i| - |\xi_i|}{\Delta|\xi_i|} \\
&= \lim_{\Delta\downarrow 0} \frac{\sqrt{(1+\Delta a_i)^2 + (\Delta\beta_i)^2} - 1}{\Delta} = a_i, \qquad (34)
\end{aligned}$$

and, hence,

$$\mathrm{Re}\,\lambda_i(A) \leqq \max_{i=1}^{n} \mathrm{Re}\,\lambda_i(A) \leqq \max_{|\xi|=1} \lambda(A; \xi, 0), \quad \forall\, i. \qquad (35)$$

By (33) and (35), we have $\mathrm{Re}\,\lambda_i(D_x f(x,p) \leqq -\alpha$ for all i, and, hence, the condition (6) holds.

2) We will show that (11) is met.

Given arbitrary $x, y \in \mathcal{X}$ such that $x \neq y$. Applying Mean-Value Theorem [6], we have

$$f(x,p) - f(y,p) = \int_0^1 D_x f(\tau x + (1-\tau)y, p) d\tau (x - y)$$

By applying above formula, properties (*iii*) and (*iv*) in Lemma 5, the conditions (13) and (14), we have

$$\begin{aligned}
\lambda(f(\cdot,p); x, y) &\leqq \int_0^1 \lambda(F(\cdot; \tau x + (1-\tau)y, p); \xi, 0) d\tau \\
&\leqq \int_0^1 \sum_{q=1}^{Q} \beta_q(\tau x + (1-\tau)y, p) \lambda(F_q; \xi, 0) d\tau \leqq -\alpha,
\end{aligned}$$

and, hence, if the condition (9) is satisfied, we have the condition (11). ■

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