2017 International Symposium on Nonlinear Theory and Its Applications,
NOLTA2017, Cancun, Mexico, December 4-7, 2017

Amplitude death in a pair of two-dimensional complex Ginzburg-Landau systems coupled by delay connections

Hakui Teki†, Keiji Konishi†, and Naoyuki Hara†

†Graduate School of Engineering, Osaka Prefecture University
1-1 Gakuen-cho, Naka-ku, Sakai, Osaka 599-8531 JAPAN
Email: konishi@eis.osakafu-u.ac.jp

Abstract—This paper deals with amplitude death in a pair of two-dimensional complex Ginzburg-Landau (CGL) systems coupled by delay connections. A linear stability analysis provides a sufficient condition for the existence of amplitude death for a no-delay connection. A systematic procedure for designing connection parameters for a delay-connection, which can induce amplitude death, is presented. These analytical results are confirmed through some numerical simulations.

1. Introduction

Nonlinear oscillators, which are coupled by diffusive connections, can cease their oscillations [1, 2]. Such cessation, which is named as amplitude death, has been intensively studied because of its potential applications in engineering [3, 4, 5]. Various types of connections inducing amplitude death have been proposed [1, 2]. Especially the two types of connections, a no-delay connection and a time delay connection, have been intensively studied due to their simplicity and nature. It is well known that the no-delay connection can induce amplitude death for non-identical oscillators, but cannot for identical oscillators. On the other hand, the time delay connection can induce amplitude death even for identical oscillators [6, 7].

In the past decades, reaction-diffusion systems have been widely studied in the field of nonlinear science [8, 9]. The complex Ginzburg-Landau (CGL) system has been considered as one of the most popular reaction-diffusion systems [9, 10], since it describes the universal dynamics around Hopf bifurcation points. As is the case with studied on coupled oscillators, nonlinear phenomena in coupled CGL systems have also been investigated [11, 12, 13]. In recent years, our previous study showed that amplitude death can occur in a pair of *one*-dimensional CGL systems coupled by the no-delay connection or the time delay connection [14]: for the non-delay connection, amplitude death can occur only if the CGL systems are not identical; for the time delay connection, amplitude death can occur even if they are identical. Furthermore, our previous study proposed a systematic procedure for designing connection parameters which can induce amplitude death.

Although our previous study would be the first attempt to show amplitude death in coupled reaction-diffusion systems, it dealt only with a pair of *one*-dimensional systems. The aim of the present paper is to investigate amplitude death in a pair of *two*-dimensional CGL systems coupled by the no-delay connection or the time delay connection. The stability of a spatially uniform steady state in the coupled systems is analyzed using linear stability analysis. The analytical results allow us to provide a systematic procedure for designing the connection parameters, which can induce amplitude death independently of its system size and boundary condition. These analytical results are confirmed through some numerical examples.

2. Coupled complex Ginzburg-Landau systems

A pair of two-dimensional CGL systems,

$$\frac{\partial W_{1,2}}{\partial t} = \left\{(1 + i\omega_{1,2}) - (1 + i\beta)\left|W_{1,2}\right|^2\right\} W_{1,2} + (1 + i\alpha)\nabla^2 W_{1,2} + U_{1,2}, \quad (1)$$

are considered, where $W_{1,2}(t, x, y) \in \mathbb{C}$ are state variables at time $t \geq 0$ and at location $(x, y) \in [0, L_x] \times [0, L_y]$. Here, α and β are the common parameters. $\nabla^2 := \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator. These systems have oscillation frequencies $\omega_{1,2} := \omega_0 \pm \Delta\omega$ with the nominal frequency $\omega_0 \in \mathbb{R}$ and the frequency difference $\Delta\omega \in \mathbb{R}$. The connection signals are given by

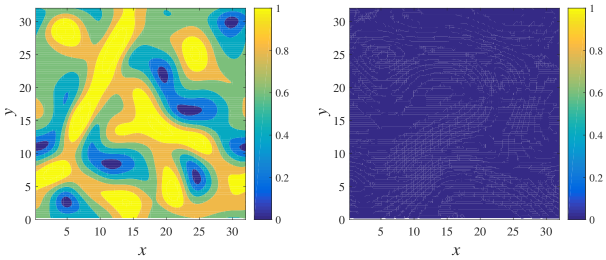
$$U_{1,2}(t, x, y) = \varepsilon \{W_{2,1}(t - \tau, x, y) - W_{1,2}(t, x, y)\}, \quad (2)$$

with the coupling strength $\varepsilon \in \mathbb{R}$ and the delay time $\tau \geq 0$.

A spatially uniform steady state of these systems without connection (i.e., $U_{1,2} \equiv 0$) is given by

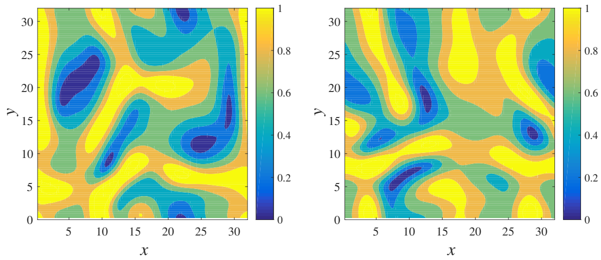
$$\begin{bmatrix} W_1(t, x, y) & W_2(t, x, y) \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \end{bmatrix}^T, \quad \forall (x, y) \in [0, L_x] \times [0, L_y]. \quad (3)$$

This state (3) exists even in coupled CGL systems (1) and (2); however, its stability depends on the connection parameters ε and τ. According to our previous study [14], it is easy to derive the characteristic equation describing the



(a) $t = 40$ (before coupling) (b) $t = 60$ (after coupling)

Figure 1: Snapshots of $|W_1|$ in a pair of non-identical CGL systems ($\Delta\omega = 4.0$) coupled by the no-delay connection with $\varepsilon = 5.0$ (a) before and (b) after couping.



(a) $t = 40$ (before coupling) (b) $t = 60$ (after coupling)

Figure 2: Snapshots of $|W_1|$ in a pair of identical CGL systems ($\Delta\omega = 0$) coupled by the no-delay connection with $\varepsilon = 5.0$ (a) before and (b) after coupling.

local stability of steady state (3),

$$F(s, \gamma, \Delta\omega, \tau) :=$$

$$\det\left(s\boldsymbol{I}_4 - \begin{bmatrix} \boldsymbol{A}(+\Delta\omega) - \varepsilon\boldsymbol{I}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}(-\Delta\omega) - \varepsilon\boldsymbol{I}_2 \end{bmatrix}\right.$$

$$\left. -\varepsilon e^{-s\tau}\begin{bmatrix} \boldsymbol{0} & \boldsymbol{R}(\omega_0\tau) \\ \boldsymbol{R}(\omega_0\tau) & \boldsymbol{0} \end{bmatrix} + \gamma\begin{bmatrix} \boldsymbol{A}(\alpha) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}(\alpha) \end{bmatrix}\right), \quad (4)$$

where

$$\boldsymbol{A}(\mathsf{x}) := \begin{bmatrix} 1 & -\mathsf{x} \\ \mathsf{x} & 1 \end{bmatrix}, \ \boldsymbol{R}(\mathsf{x}) := \begin{bmatrix} \cos\mathsf{x} & \sin\mathsf{x} \\ -\sin\mathsf{x} & \cos\mathsf{x} \end{bmatrix}. \quad (5)$$

The parameter γ is given by $\gamma := k_x^2 + k_y^2$, where $k_x, k_y \in \mathbb{R}$ are the wave numbers of x and y, respectively.

3. Stability analysis and numerical examples

In this section, we analyze the stability of steady state (3) with the no-delay connection ($\tau = 0$) or the time delay connection ($\tau > 0$), and then confirm the analytical results through some numerical examples.

3.1. No-delay connection

In accordance with our previous study [14], we can easily obtain the following stability condition from the characteristic function $F(s, \gamma, \Delta\omega, 0)$:

$$\varepsilon > 1, \ \Delta\omega > \sqrt{2\varepsilon - 1}. \quad (6)$$

The uniform steady state (3) is stable for any common parameter α and β, for any boundary condition, and for any system size $L_{x,y}$ if and only if ε and $\Delta\omega$ satisfy condition (6). It should be noted that this analytical result for the two-dimensional CGL systems is equivalent to that for the one-dimensional CGL systems our previous study [14] dealt with.

Let us check our analytical results on numerical simulations. Throughout this paper, we use the explicit Euler method with a time step of $\Delta t = 1 \times 10^{-3}$ and 64×64 space mesh points, and employ the periodic boundary conditions. The parameters in the CGL systems are set to $\alpha = 3.00$ and $\beta = -1.2$, which satisfy the Benjamin-Feir criterion $\alpha\beta < -1$, and $\omega_0 = 6.0$. Now we set $\varepsilon = 5.0$ and $\Delta\omega = 4.0$, which satisfy condition (6). Figure 1 shows the snapshots[1] of $|W_1|$ in a pair of non-identical CGL systems ($\Delta\omega = 4.0$) coupled by the no-delay connection before coupling and after coupling[2]. It can be seen that these systems behave chaotically before coupling, and then converges on uniform steady state (3) after coupling. This fact suggests that amplitude death can occur in coupled two-dimensional CGL systems. Here we assume the identical CGL systems: $\Delta\omega = 0$ never satisfies condition (6). The snapshots of $|W_1|$ in a pair of identical CGL systems before coupling and after coupling are demonstrated in Figure 2. We see that these CGL systems behave chaotically even after coupling: amplitude death does not occur in coupled identical CGL systems.

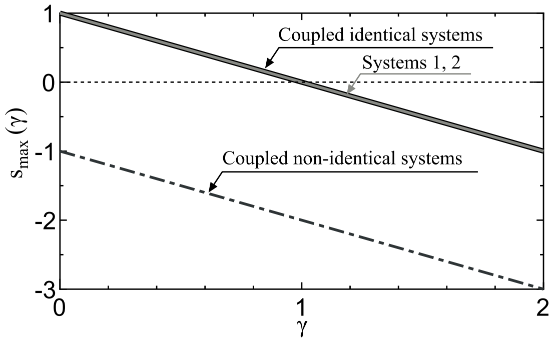
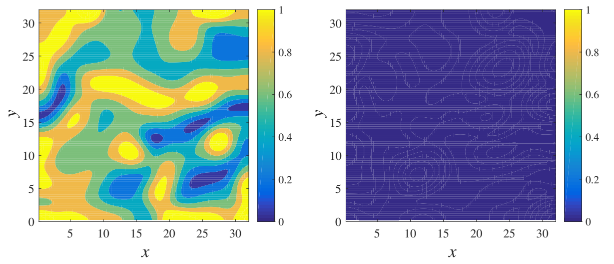


Figure 3: Real part of the dominant roots, $s_{\max}(\gamma)$, of the characteristic equations for isolated systems 1, 2, coupled identical systems, and coupled non-identical systems as a function of $\gamma \geq 0$.

The above numerical results are now explained by using

[1] Yellow corresponds to $|W_1| = 1$ and blue to $|W_1| = 0$.

[2] We have confirmed that the snapshots of $|W_2|$ are similar to that of $|W_1|$ throughout this paper.



(a) t = 40 (before coupling)   (b) t = 100 (after coupling)

Figure 4: Snapshots of $|W_1|$ in a pair of identical CGL systems coupled by the time delay connection with $\varepsilon = 5.0$ and $\tau = 0.25$ (a) before and (b) after coupling.

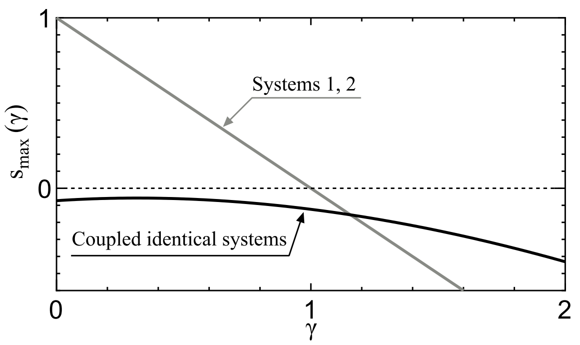


Figure 5: Real part of the dominant roots, $s_{\max}(\gamma)$, of the characteristic equations for isolated systems 1, 2, and coupled identical systems ($\alpha = 3.00$) as a function of $\gamma \geq 0$.

real part of dominant roots of $F(s, \gamma, \Delta\omega, 0) = 0$:

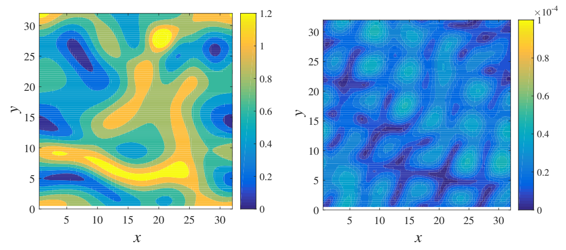
$$s_{\max}(\gamma) := \max_{i \in \{1,\ldots,n\}} \mathrm{Re}[s_i(\gamma)], \tag{7}$$

where $n = 4$. Figure 3 shows that $s_{\max}(\gamma)$ of isolated systems 1, 2, and that of coupled identical systems are equivalent. This fact indicates that the positive real part cannot be suppressed by using the no-delay connection. On the other hand, $s_{\max}(\gamma)$ of coupled non-identical systems is less than zero for all $\gamma \geq 0$. This suppression suggests that the uniform steady state (3) is stable for any common parameter α and β, for any boundary condition, and for any system size $L_{x,y}$.

3.2. Time delay connection

We have seen from stability condition (6) that the no-delay connection never induces the stabilization of uniform steady state (3) in identical CGL systems ($\Delta\omega = 0$). This subsection deals with stability of steady state (3) in identical CGL systems ($\Delta\omega = 0$) coupled by the time delay connection ($\tau > 0$). The characteristic function (4) with $\Delta\omega = 0$ is given by

$$F(s, \gamma, 0, \tau) := F_+(s, \gamma) F_-(s, \gamma), \tag{8}$$



(a) t = 150 (before coupling)   (b) t = 1000 (after coupling)

Figure 6: Snapshots of $|W_1|$ in a pair of identical CGL systems ($L_{x,y} = 32$) coupled by the time delay connection with $\varepsilon = 5.0$ and $\tau = 0.25$ (a) before and (b) after coupling.

$$F_\pm(s, \gamma) := \det\left[(s-1)\boldsymbol{I}_2 + \gamma \boldsymbol{A}(\alpha) + \varepsilon\left\{\boldsymbol{I}_2 \pm e^{-s\tau}\boldsymbol{R}(\omega_0\tau)\right\}\right]. \tag{9}$$

As is the case with our previous study [14], a systematic procedure for designing ε and τ to induce the stabilization of steady state (3) can be easily derived from the characteristic function (8).

(Step 0) $\omega_0 \neq 0$ and α are obtained.

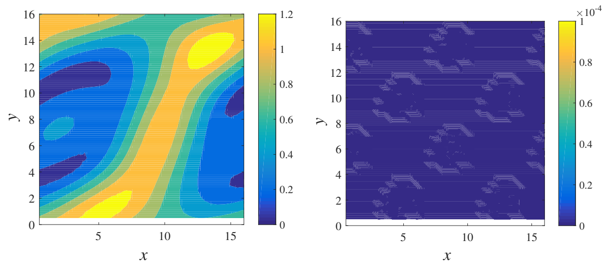
(Step 1) Choose ε and τ such that $f_\pm(s, 0)$ is stable, where

$$f_\pm(s, \gamma) := s - i\omega_0 - 1 + (1 + i\alpha)\gamma + \varepsilon\left(1 \pm e^{-s\tau}\right). \tag{10}$$

(Step 2) The solutions of $f_+(i\lambda_I, \gamma) = 0$ and that of $f_-(i\lambda_I, \gamma) = 0$ are not within the range $\gamma \in [0, 1]$ on the $\gamma - \lambda_I$ plane.

The steady state (3) is stable for any parameter β, for any boundary condition, and for any system size $L_{x,y}$ if and only if ε and τ go through these steps. Remark that these steps are the same as those for coupled one-dimensional CGL systems our previous study [14] dealt with.

We now check our analytical results on numerical simulations. The parameters in the CGL systems are set to $\alpha = 3.00$, $\beta = -1.2$, and $\omega_0 = 6.0$. For (Step 0), ω_0 and α are obtained. For (Step 1), $\varepsilon = 5.0$ and $\tau = 0.25$ are chosen such that functions (10) are both stable. For (Step 2), we numerically confirm that the solutions of $f_+(i\lambda_I, \gamma) = 0$ and those of $f_-(i\lambda_I, \gamma) = 0$ are not within the range $\gamma \in [0, 1]$. Figure 4 shows the snapshots of $|W_1|$ in a pair of identical CGL systems coupled by the time delay connection before coupling and after coupling. These systems behave chaotically before coupling, and then amplitude death occurs after coupling. Let us check real part of dominant roots of $F(s, \gamma, 0, \tau) = 0$, which is defined by Eq. (7) with $n \to +\infty$. Figure 5 shows $s_{\max}(\gamma)$ of isolated systems 1, 2 and coupled identical systems. As can be seen that the time delay connection reduces the positive real part under zero. We notice that all the analytical and numerical results mentioned



(a) t = 150 (before coupling) (b) t = 1000 (after coupling)

Figure 7: Snapshots of $|W_1|$ in a pair of identical CGL systems ($L_{x,y}$ = 16) coupled by the time delay connection with ε = 5.0 and τ = 0.25 (a) before and (b) after coupling.

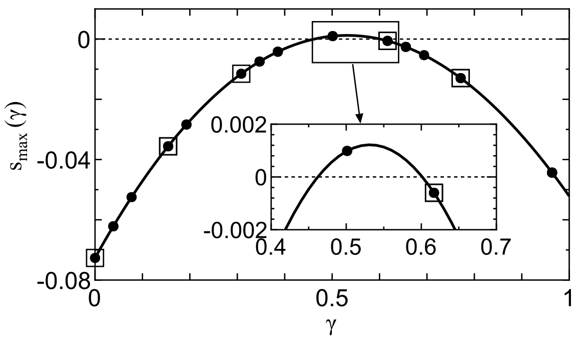


Figure 8: Real part of the dominant roots, $s_{\max}(\gamma)$, of the characteristic equations for coupled identical systems (α = 4.05) as a function of $\gamma \geq 0$. The symbols • and □ denote the wave numbers of $L_{x,y}$ = 32 and $L_{x,y}$ = 16, respectively.

above have been agreed.

Let us consider behavior of coupled identical systems (α = 4.05) with ε and τ which do not go through (Step 0)-(Step 2). Now we fix ε = 5.0 and τ = 0.25 which satisfy (Step 1), but not (Step 2). As shown in Fig. 6, amplitude death in the time-delay-coupled identical systems with $L_{x,y}$ = 32 does not occur[3]. On the other hand, for another size with $L_{x,y}$ = 16, amplitude death occurs as shown in Fig. 7. These numerical results show that, for ε and τ which do not go through (Step 0)-(Step 2), the existence of amplitude death depends on the system size. Figure 8 shows $s_{\max}(\gamma)$ of coupled identical systems. It can be seen that $s_{\max}(\gamma) > 0$ holds for $\gamma \in (0.47, 0.60)$. Here $s_{\max}(\gamma)$ at the wave numbers of system sizes $L_{x,y}$ = 32 and $L_{x,y}$ = 16 corresponding to Figs. 6 and 7 are plotted with solid circles and open squares, respectively. There exits one positive solid circle at γ = 0.5, but there is no positive open square. This fact indicates that steady state (3) in the coupled systems with $L_{x,y}$ = 32 is unstable at wave number γ = 0.5, but that with $L_{x,y}$ = 16 is stable (i.e., amplitude death can occur).

[3] Although amplitude of $|W_1|$ is small (see the order of color bar in Fig. 6(b)), amplitude death does not occur.

4. Conclusion

This paper investigated amplitude death in two-dimensional CGL systems coupled by the no-delay connection or the time delay connection. We showed the sufficient stability condition for the no-delay connection. The systematic procedure for design of the connection parameters inducing amplitude death for the time delay connection was provided. These analytical results were confirmed with numerical examples.

Acknowledgments

The present research was partially supported by JSPS KAKENHI (26289131).

References

[1] G. Saxena *et al.*, Phys. Rep., vol. 521, pp. 205–228, 2012.

[2] A. Koseska *et al.*, Phys. Rev. Lett, vol. 111, p. 024103, 2013.

[3] S. Huddy and J. Skufca, IEEE Trans. Power Electronics, vol. 28, pp. 247–253, 2013.

[4] D.Q. Wei *et al.*, IEEE Trans. Circuit Sys., vol. 60, pp. 692–696, 2013.

[5] T. Biwa *et al.*, Phys. Rev. Appl., vol. 3, p. 034006, 2015.

[6] D. Reddy *et al.*, Phys. Rev. Lett., vol. 80, pp. 5109–5112, 1998.

[7] D. Reddy *et al.*, Physica D, vol. 129, pp. 15–34, 1999.

[8] J.D. Murray, Mathematical Biology II, Springer-Verlag, 2003.

[9] M. Cross and H. Greenside, Pattern Formation and Dynamics in Nonequilibrium Systems, Cambridge University Press, 2009.

[10] I.S. Aranson and L. Kramer, Rev. Mod. Phys., vol. 74, pp. 99–143, 2002.

[11] S. Boccaletti *et al.*, Phys. Rev. Lett., vol. 83, pp. 536–539, 1999.

[12] J. Bragard *et al.*, Int. J. of Bifurcation and Chaos, vol. 11, pp. 2715–2729, 2001.

[13] H. Nie *et al.*, Phys. Rev. E, vol. 84, p. 056204, 2011.

[14] H. Teki *et al.*, Phys. Rev. E, vol.95, p. 062220, 2017.