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Bifurcations of a Dynamical System Defined by PMART

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Abstract—In this paper, We investigate dynamical behavior of an iterative image reconstruction technique called PMART(Power Multiplicative Algebraic Reconstruction Technique) used for the X-ray computed tomography(CT). This technique can obtain an accurate image by iterating an updating scheme for simultaneous equations even though the iteration consumes much time at first. In addition, the updating schemes fails sometimes when the parameters of PMART are provided inappropriately. Recently this method is revised since the scheme can be solved within a reasonable processing time with high-speed computers. In this paper, we analyze the PMART updating scheme as a bifurcation problem of the dynamical system. We would like to insist that above failure is caused by bifurcations of the system. We develop a method for solving this bifurcation problem, and as a result, we obtain bifurcations diagrams of the fixed or periodic points. This result suggests us parameter regions and an initial value set in which the PMART can operate normally.

1. Introduction

As a popular image reconstruction method for X-ray computed tomography (CT), the filtered back-projection method is widely used since its real-time processing ability, however, this may contain some line-like noise called artifact. Sometimes it is hard to remove this noise perfectly without artificial data manipulation by nonlinear filters.

In contrary, Iterative image reconstruction method[1], which forms a difference dynamical system, is in the spotlight in recent years. ART, MART, PMART[3] have been proposed. Although this method can result a fine image, but it consumes much time to converge its repetitive computation and requires much memory. Nowadays these difficulties are being overcome as computers are progressing.

In this paper, we investigate dynamical properties of PMART. This forms a closed form of a nonlinear difference equation, and it is differentiable. The system exhibits some typical bifurcation phenomena such as double Neimark-Sacker bifurcation, tangent bifurcation, and so on. by changing parameter values during repetitive computation. We develop a computation method of bifurcation parameter values in PMART dynamics by using Newton's method. As an illustrative example, bifurcation and nonlinear characteristics of a 2×2 image is investigated.

2. Image Reconstruction Method

Figure 1 shows a principle of X-ray photograph and X-ray CT[4]. Both methods utilize X-rays which penetrate the object. In the case of the X-ray photograph, the image can be directly obtained as an exposed film by this penetrated X-rays. On the contrary, X-ray CT is designed for observing sliced images of an object. The detectors record the intensity of the penetrated X-rays passing through many internal organs. An appropriate method is required to reconstruct the sliced image from a set of detected values which all are perpendicular to the object.

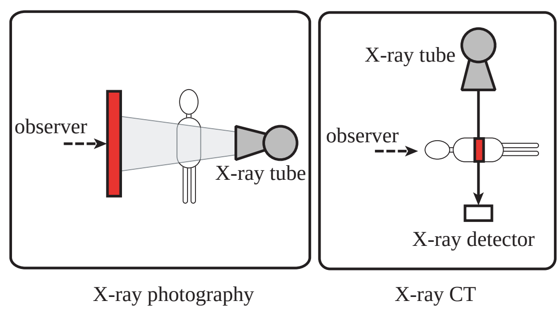

Figure 1: Principle of X-ray photograph and X-ray CT

The image reconstruction method used for X-ray CT is classified into two major methods; the analytical reconstruction method and the iterative reconstruction method. Examples of the analytical reconstruction method include the filtered back-projection method which is the current mainstream. This technique can acquire a re-constitution image quickly. However, there is the case that a thread noise called artifacts occurred by used projection data.

The iterative image reconstruction technique method is an iterative algorithm which has been proposed at the beginnings of CT development. Let us assume an arbitrary initial image. This technique computes errors between projection data calculated by this image and detected values, and update/modify the image data by using an updating scheme with appropriate evaluation functions. A high-resolution reconstructed image is obtained by this scheme, however, it becomes a time-consuming algorithm for large size images. Recently, a quick convergence algorithm called PMART attracts attentions because the algorithm can be operated within a reasonable computation time.

3. Iterative Image Reconstruction Technique

The basic iterative image reconstruction technique algorithm can be generally written as the form:

$$\boldsymbol{x}_{n+1} = \boldsymbol{f}(\boldsymbol{x}_n), \quad n = 1, 2, \ldots \tag{1}$$

where $\boldsymbol{x}_{n+1}$ and $\boldsymbol{x}_n$ are the successive estimations at n and $n+1$. It is necessary to have the character that $\boldsymbol{x}_{n+1}$ converges in true pixel value vector $\boldsymbol{x}^*$ (true value what we want to obtain) to approve the function $\boldsymbol{f}$. There is a technique that called ART(Algebraic Reconstruction Technique)[2] that is the basic iterative image reconstruction method. Hereafter we explain briefly this method.

We assume that an image(object) $x = (x_1, x_2, x_3, x_4)$ of 2×2 shown in Fig. 2, and each $x_k, k = 1, 2, 3, 4$ is unknown. X-rays pass through the object in various directions, and we name the projection operator of the i-th direction as p^i. Totally operators are as six vectors p^1–p^6 shown in Fig. 2. Thereby the observed value (projection) of the i-th direction can be expressed by $p^i\boldsymbol{x}$. For instance, the first observed value is as $p^1\boldsymbol{x} = x_1 + x_3$.

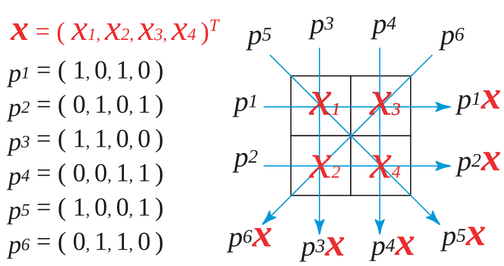

Figure 2: The definition of the projection

The ART algorithm can be written as the following iterative dynamical system:

$$x_j^{i+1} = x_j^i + p_j^i \left(\frac{p^i\boldsymbol{x}^* - p^i\boldsymbol{x}}{N^i} \right), \quad i = 1, 2, \ldots, 6. \tag{2}$$

where i is the i-th projection, j is the j-th pixel of the image, N^i is the number of the projected pixels, $p^i\boldsymbol{x}^*$ and $p^i\boldsymbol{x}$ are the projection values of the phantom image (an object that should be actually desired) and the projections of the reconstruction image, respectively. We only can observe projection values $p^i\boldsymbol{x}$ and $p^i\boldsymbol{x}^*$ by calculation and measuring.

Let us assume the initial value of the reconstruction image $\boldsymbol{x} = (1, 2, 3, 4)$ and the projection of the phantom $p^1\boldsymbol{x}^*$ gives 12. Then the i-th projection $p^1\boldsymbol{x} = (x_1 + x_3)$ gives 4. We update the reconstruction image $\boldsymbol{x}$ by appropriate method to output the same values for $p^1\boldsymbol{x}^*$. If differences $p^i\boldsymbol{x}^* - p^i\boldsymbol{x}$ for all i are vanished, we obtain the correct phantom image finally.

4. PMART

In this section, we briefly introduce PMART as a powerful iterative image reconstruction technique. We consider a J−dimensional discrete dynamical system $\boldsymbol{x}(k+1) = \boldsymbol{f}(\boldsymbol{x}(k)), \quad k = 1, \ldots$, or a map defined by the following equation:

$$\boldsymbol{f} : \boldsymbol{R}^J \to \boldsymbol{R}^J; \quad \boldsymbol{x} \mapsto \boldsymbol{f}(\boldsymbol{x}) \tag{3}$$

where $\boldsymbol{x}(k) = (x_1, x_2, \cdots x_J)^T$ is a state vector in R^J. The PMART algorithm can be written in the mapping form with the following elements:

$$\boldsymbol{f}_j = \boldsymbol{f}_j^I \circ \boldsymbol{f}_j^{I-1} \circ \ldots \circ \boldsymbol{f}_j^1, \quad j = 1, \ldots, J. \tag{4}$$

with i-th submap

$$\boldsymbol{f}_j^i = x_j \left(\frac{p^i\boldsymbol{x}^*}{p^i\boldsymbol{x}} \right)^{\gamma p_j^i}, \quad i = 1, 2, \ldots, I. \tag{5}$$

where parameter γ is an arbitrary real number, and it controls acceleration of convergence of this iterative method.

Figures 3 and 4 are examples of image reconstruction by ART and PMART respectively. Projections from 396 directions give a 16 × 16 pixel image from a simplified Shepp-Logan head phantom. We try 5 iterations for this image.

A white pixel has a high intensity of CT value, probably it depicts bones, while, a black pixel is low, and it depicts an empty space. The initial values of reconstruction image is all zero. In each figure, the most left image is phantom image, The most right image is reconstruction image after five iterations. In the ART, it is not enough to reconstruct the phantom image with only five iterations, however, PMART gives a reasonable result within three or four iterations in this example. We may conclude that PMART converges faster than ART because of its acceleration effect of the power law.

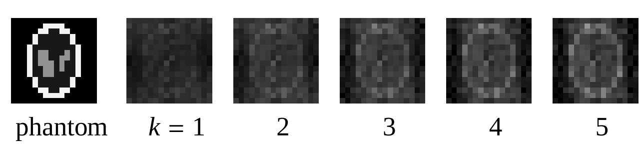

Figure 3: ART

However, in the case of $\gamma \sim 2$ in PMART, the scheme does not converge to the ture image sometimes. It depends on the initial image, see Fig.5. And, the image reconstruction fails so that the solution emanates. In this case, the iterative scheme converges to a false image $\boldsymbol{x}_0$ (false fixed

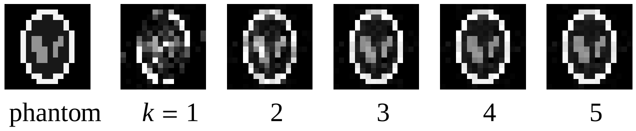

Figure 4: PMART

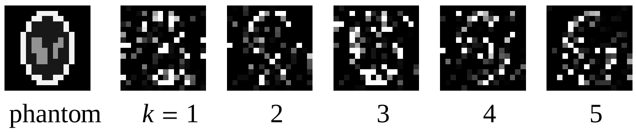

Figure 5: PMART(γ=2.0)

point) even though there exist also the phantom image $\boldsymbol{x}^*$ (true fixed point). Moreover, double Neimark-Sacker bifurcation occurs in the case of $\gamma = 2.0$. Thus an invariant closed curve(ICC), $\boldsymbol{x}^* = (X_1, X_2, X_3, X_4) = (5, 7, 6, 2)$, and $\boldsymbol{x}_0 = (5.3, 7.2, 5.7, 1.8)$ are observed at this value of γ, see Fig.6. The symbol $\times$ depicts the phantom. The fixed points disappear by the tangent bifurcation as the value of γ decreases.

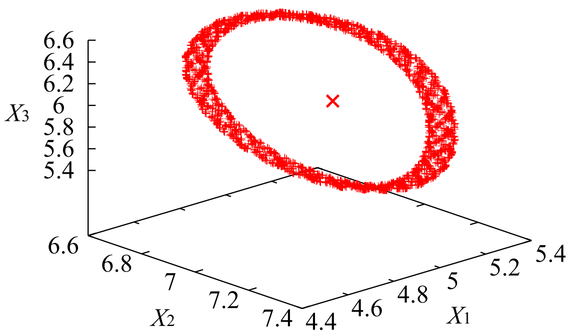

Figure 6: Invariant closed curve with (γ=2.0)

To analyze such generation of fixed points and ICC, we formulate the iterative scheme PMART as an bifurcation problem, i.e., we try to explain the dynamical behavior of the system by bifurcation diagrams illustrating topological properties of the limit sets. At first, by this research, We analyze PMART with $I = 6$, $J = 4$ for the 2×2 pixel image.

5. Locating the bifurcation curves

We briefly show a trace procedure of the bifurcation curves below. At first, We think about Eq.(6) including parameter λ of system in Eq.(3) which We showed in a foregoing paragraph.

$$T : \boldsymbol{R}^J \to \boldsymbol{R}^J; \quad \boldsymbol{x} \mapsto T(\boldsymbol{x}, \lambda) \tag{6}$$

When we change a parameter of system, there is the case that a topological property of the fixed point $\boldsymbol{x}_0^* \in \boldsymbol{R}^J$ might change, and this is called a bifurcation.

In the case where a bifurcation of a fixed point is occurred in $\lambda = \lambda_0$, then λ_0 is called a bifurcation point. We can precisely obtain the bifurcation points by solving the simultaneous equations of the following Eq.(7) and Eq.(8) because the character of the fixed points is related to the eigenvalue. A fixed point of Eq.(6) can be obtained by Eq.(7).

$$\boldsymbol{f}(\boldsymbol{x}_0^*, \lambda) := T(\boldsymbol{x}_0^*, \lambda) - \boldsymbol{x}_0^* = \boldsymbol{0}. \tag{7}$$

Moreover, the characteristic equation is given as follows:

$$g(\boldsymbol{x}_0^*, \lambda) := \det\left(\mu I - \frac{\partial T_\lambda}{\partial \boldsymbol{x}_0}(\boldsymbol{x}_0^*)\right) = 0. \tag{8}$$

A topological character of a fixed point changes when characteristic multiplier μ of Eq.(8) crosses the unit circle of the complex plane. For instance, the tangent bifurcation happens when one of the eigenvalues exceeds $\mu = 1$. At this time, the fixed point disappears, and the orbit may jump to another position rapidly (jump phenomenon). Moreover, various bifurcation phenomena exist such as the period-doubling bifurcation($\mu = -1$) and Neimark-Sacker bifurcation ($|\mu| = 1$). Eqs. (7) and (8) are defined by $\boldsymbol{F} \in \boldsymbol{R}^5$, and these equations are merged together by Eq.(9) as the form:

$$\boldsymbol{F}(\boldsymbol{x}_0^*, \lambda) = (\boldsymbol{f}(\boldsymbol{x}_0^*, \lambda), \boldsymbol{g}(\boldsymbol{x}_0^*, \lambda))^T. \tag{9}$$

By using Newton's method for Eq.(10), We can trace the bifurcation curves.

$$\begin{cases} \boldsymbol{x}^{(k+1)} & = & \boldsymbol{x}^{(k)} + h \\ F'(\boldsymbol{x}^{(k)}) & = & -F(\boldsymbol{x}^{(k)}) \end{cases} \tag{10}$$

The method requires the 1st and 2nd variational equations about initial values X_i and parameter γ to solve Eq. (10). Besides, determinants and its derivatives for parameters appearing in this method for parameters forms very complicated. For example, the 1st differentiation of $\boldsymbol{f}$ with respect to $\boldsymbol{x}$ is written as Eqs. (11) and (12).

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{f}^I}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{f}^{I-1}}{\partial \boldsymbol{x}} \cdots \frac{\partial \boldsymbol{f}^1}{\partial \boldsymbol{x}}. \tag{11}$$

$$\frac{\partial \boldsymbol{f}^i}{\partial \boldsymbol{x}} = \mathrm{diag}_j \left\{ \left(\frac{p^i \boldsymbol{x}^*}{p^i \boldsymbol{x}} \right)^{\gamma p^i_j} \right\} \left(E - \frac{\gamma \cdot p^i \cdot p^{iT}}{p^i \boldsymbol{x}} \mathrm{diag}_j\{x_j\} \right) \tag{12}$$

for $i = 1, 2, \ldots, 6$, where E denotes the 4×4 identity matrix. Moreover, the parameter used when the bifurcation curves obtained is five parameters of γ and the each element (X_1, X_2, X_3, X_4) of the phantom image $\boldsymbol{x}^*$. Three parameters are fixed among these. Two parameters of the remainder are assumed to be an unknown number, and the bifurcation curves is traced. We consider that it chiefly depends on γ by considering section 4, and the character of PMART changes. We trace the bifurcation curves mainly on γ in this research.

6. Bifurcation diagram

Through this paper, three parameters $X_2 = 7, X_3 = 6, X_4 = 2$ are fixed. Figure7 is a bifurcation diagram in γ-X_1 space. Symbols I and G shows the period-doubling and tangent bifurcations, respectively. The superscript shows the number of period and the subscript shows distinction of bifurcation curves. Figure 8 is an enlarged illustration of the part which it surrounded with a square in Fig.7. From these curves, We can divide the whole bifurcation diagram into five domains symbolized by A to E.

In all domains, There is certain a initial value set from which the PMART converges to the phantom image. The most important result of this work is the following fact: the iterative scheme fails if the γ and one of the initial values X_1 are chosen in the upper region bounded by G^1 and G^2. Figures 9 illustrates a topological classification of the system. And they show a change of a topological property by a parameter changes.

We consider about the case where a fixed point is given as an initial value in domain A. When we let parameter X_1 increase, it becomes two-periodic point by period-doubling bifurcation I_1^1 in domain B. Furthermore, it changes at a fixed point of a false value in domain C by the tangent bifurcation G^2 when we let X_1 increase. In case that we let parameter X_1 decrease in domain C, a fixed point changes into two-periodic point by I_2^1 in domain B.

In domain E, any orbit converges to the true value. In addition, it converges to the true value by the tangent bifurcation in domain D when it decreased γ from domain A.

Next, we consider about the case where a two-periodic point is given as an initial value in domain A. It changes into the fixed point of a false value by the tangent bifurcation in domain C when we let parameter X_1 increase. In domain E, it converges to the true value. In addition, a two-periodic point converges to the true value in domain E by G^2 when We let γ decrease in domain A. There is only the fixed point that is stability in domain E. Rather than these things, the image reconstruction by the PMART is enabled certain by setting a parameter equivalent to domain E.

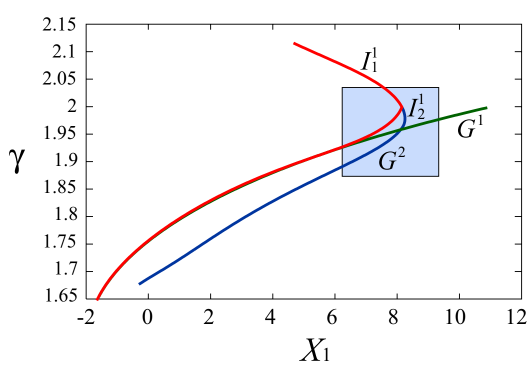

Figure 7: Bifurcation diagram

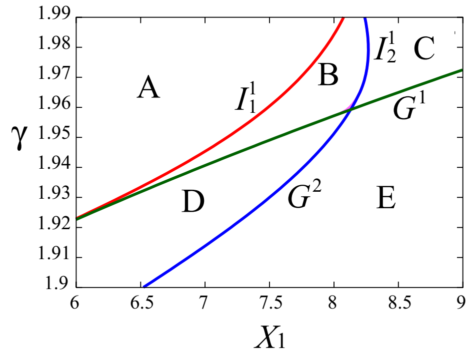

Figure 8: Enlarged diagram of Fig. 7.

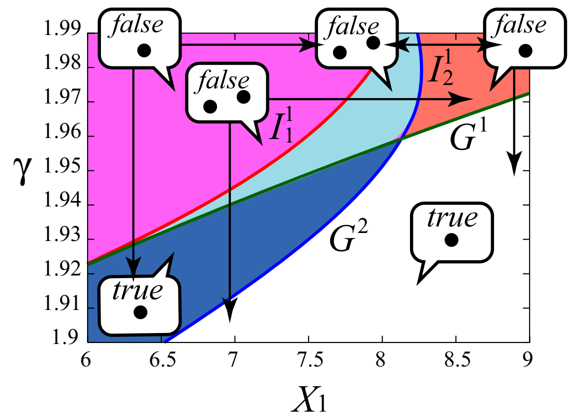

Figure 9: Topological classification of fixed points

7. Conclusion

In this paper, we obtain bifurcation diagrams of PMART. We develop a calculation method for obtaining bifurcation curves, and show some their diagrams. We assert that the failure of the PMART is caused by bifurcations. From these bifurcation diagrams, one can know a certain parameter range in which PMART is able to operate normally.

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