2017 International Symposium on Nonlinear Theory and Its Applications, NOLTA2017, Cancun, Mexico, December 4-7, 2017

Generalized multi-synchronization of chaotic systems via dynamical control laws: stability of synchronization manifold

Christopher D. Cruz-Ancona†, Rafael Martínez-Guerra† and Claudia A. Pérez-Pinacho†

†Department of Automatic Control, CINVESTAV-IPN,
Ave. IPN 2508, San Pedro Zacatenco, Mexico City 07360, Mexico
Email: ccruz@ctrl.cinvestav.mx, rguerra@ctrl.cinvestav.mx, caperez@ctrl.cinvestav.mx

Abstract—Within a differential algebraic framework, this paper studies the synchronization phenomena for networks of strictly different nonlinear chaotic systems, i.e., generalized multi-synchronization (GMS). In this case, by allowing any type of interplay between slave systems in a master multi-slave topology, a dynamical control law with diffusive coupling terms is designed for each slave system to synchronize the whole network. Moreover, with the premise that differential algebraic techniques allows us to completely characterize its synchronization manifold, we present some preliminary results on stability of synchronization manifolds. Finally, the effectiveness of the approach is shown in numerical simulations.

1. Introduction

As seen in nature, common dynamical behavior is expected within groups of interacting units, i.e., coupled oscillators [1]. When this correlated motion occurs, synchronization is achieved for all nodes in the network. It is currently known that it could happen not only for groups of limit cycle oscillators but in the case of chaotic systems as well. The latter case is an active research area since its introducction by Pecora and Carroll [2] with potential applications in secure communications. Recently, a great deal of attention has been given towards the study of networks of interacting systems with distinct dynamics [3, 4, 5], some of those efforts are given in terms of stability in a practical sense.

In this paper, we study a regimen of synchronization of multiple interacting units of chaotic systems in a master slave configuration so-called GMS (with any type of interplay[1] between slave systems). This phenomena occurs when the coupled units are synchronized by means of a functional mapping that relates the trajectories of each slave with its master. GMS problem is twofold: first, find that mapping which also describes the synchronization manifold; and second, verify stability of synchronization manifold. Definition of synchronization manifold is a key element to describe the synchronous behavior of the network as a whole. Its stability determines if a stable synchronous behavior can be achieved [6].

[1]See [5] for a GMS regime with no interacting slave nodes.

It is clear that finding a synchronization manifold depends on the differences between the dynamics of individual nodes in the network. For networks of identical units (homogeneous networks) corresponds a trivial synchronization manifold, this is given as the equality of the states; otherwise, for heterogeneous networks, it does not necessarily exists. Differential algebraic techniques let us: completely characterize the synchronization manifold in the GMS problem (even for heterogeneous networks); and design dynamical control laws to synchronize the whole heterogeneous network in this generalized sense.

Notation $\otimes$ denotes the Kronecker product of matrices, Matrix $0_{\mu\times\nu} \in \mathbb{R}^{\mu\times\nu}$ represents a matrix with all entries equal to zero, Matrix $I_\nu \in \mathbb{R}^{\nu\times\nu}$, is the identity matrix, $A_\nu = \begin{pmatrix} 0_{(\nu-1)\times 1} & I_{\nu-1} \\ 0 & 0_{1\times(\nu-1)} \end{pmatrix} \in \mathbb{R}^{\nu\times\nu}$, $B_\nu = \begin{pmatrix} 0_{(\nu-1)\times 1} \\ 1 \end{pmatrix} \in \mathbb{R}^\nu$ represent a pair of matrix and vector for a system in canonical form, Vector $\mathbf{1}_\nu = (1, ..., 1)^T \in \mathbb{R}^\nu$ denotes a ν-dimensional column vector with all elements being 1; $\mathcal{G}_\nu \triangleq (\mathcal{V}_\nu, \mathcal{E}_\nu, \mathcal{A}_\nu)$ denotes a directed graph with a set of nodes $\mathcal{V}_\nu = \{1, ..., \nu\}$, a set of edges $\mathcal{E}_\nu = \mathcal{V}_\nu \times \mathcal{V}_\nu$, and adjacency matrix $\mathcal{A}_\nu = [a_{ij}] \in \mathbb{R}^{\nu\times\nu}$ where $a_{ij} = 1$ if $(j, i) \in \mathcal{E}_\nu$ and $a_{ij} = 0$ otherwise; $\mathcal{L}_\nu = [l_{ij}] \in \mathbb{R}^{\nu\times\nu}$ denotes a nonsymmetrical Laplacian matrix associated with $\mathcal{G}_\nu$ where $l_{ij} = \sum_{j=1, i\neq j}^{\nu} a_{ij}$ if $i, j = 1, ..., \nu$ and $l_{ij} = -a_{ij}$ otherwise; $\|\cdot\|_p$ represents the usual vector p-norm and $\|x\|_{\mathcal{M}} \triangleq \inf_{\hat{x}\in\mathcal{M}} \{\|x - \hat{x}\|_2\}$ denotes the point x to set $\mathcal{M}$ distance; a positive definite symmetric matrix P is denoted by $P > 0$ with $\lambda_{max}(P) > \lambda_{min}(P) > 0$ as the maximum and minimum eigenvalue of P, respectively; $[\partial/\partial x]f$ denotes the Jacobian matrix of a vector valued function f.

2. Preliminaries

Consider a group of $N + 1$ decoupled chaotic systems described by individual dynamics,

$$\begin{aligned} \dot{x}_i &= f_i(x_i, u_i) \\ \dot{y}_i &= h_i(x_i, u_i) \end{aligned} \tag{1}$$

where $x_i = (x_{i1}, ..., x_{in}) \in \Omega_i \subset \mathbb{R}^n$, $u_i \in \mathbb{R}$ and $y_i \in \mathbb{R}$ denote the state, input and output of the $i-th$ unit, $i = 1, ..., N + 1$, respectively. In the rest of this Section, some important elements of differential algebra in control theory are given (for further details, see [7]).

Definition 1 *A dynamics is a finitely generated differential algebraic extension $\mathcal{D}/k\langle u_i\rangle$ of the field $k\langle u_i\rangle$, i.e.,* $\operatorname{diff}\operatorname{tr}\operatorname{d}^\circ \mathcal{D}/k\langle u_i\rangle = 0$.

Definition 2 *Let $\{u_i, y_i\}$ be a subset of $\mathcal{D}$ in a dynamics $\mathcal{D}/k\langle u_i\rangle$. An element $x_{ij} \in \mathcal{D}$ is called observable with respect to $\{u_i, y_i\}$ if it is algebraic over $k\langle u_i, y_i\rangle$, i.e., if x_{ij} can be expressed as an algebraic function of the components of $\{u_i, y_i\}$ and a finite number of their time derivatives.*

The Theorem of the differential primitive element states that there exists a single element $\delta \in \mathcal{D}$, which is a differential primitive element, such that $\mathcal{D} = k\langle\delta\rangle$, i.e., $\mathcal{D}$ is differentially generated by k and δ.

Definition 3 *A system is Picard-Vessiot (PV) if and only if the $k\langle u\rangle$-vector space generated by the derivatives of the set $\left\{y_i^{(\mu)}, \mu \geq 0\right\}$ has finite dimension, with y_i as differential primitive element.*

Assume that it is possible to rewrite systems (1) into their GOCF:

Lemma 1 ([5]) *Consider an observable system (1) for i fixed and choose its differential primitive element as:*

$$y_i = \sum_{j=1}^{n} \alpha_j x_{ij} + \sum_{k=1}^{m} \beta_k u_{ik}, \quad \alpha_j, \beta_k \in k\langle u_i\rangle \tag{2}$$

where $u_i = (u_{i1}, ..., u_{im})^T$. System (1) is PV if and only if it can be transformable to a Generalized Observability Canonical Form (GOCF):

$$\begin{aligned} \dot{\eta}_i &= A_n\eta_i + B_n\left(g_i(U_i, \eta_i) + \bar{u}_i\right) \\ \dot{U}_i &= A_\gamma U_i + B_\gamma \bar{u}_i \end{aligned} \tag{3}$$

with $\eta_i := \phi_{u_i}(x_i) = (y_i, \dot{y}_i, ..., y_i^{(n-1)})^T \in \mathbb{R}^n$ as the control dependent nonlinear transformation and; $U_i = (u_i, \dot{u}_i, ..., u_i^{(\gamma-1)})^T \in \mathbb{R}^\gamma$, $\bar{u}_i \in \mathbb{R}$ as dynamical control law as a chain of integrators for input u_i; obtained from the differential primitive element and its $n-1$ time derivatives. □

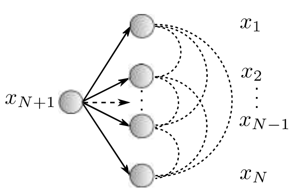


Figure 1: Directed spanning tree $\mathcal{G}_{N+1} = (\mathcal{V}_{N+1}, \mathcal{E}_{N+1}, \mathcal{A}_{N+1})$ in GMS.

3. Main result

Assume conditions of Lemma 1 fulfills. Moreover, consider the spanning tree $\mathcal{G}_{N+1}$ in Figure 1, it models the interaction of the individual units (1) and (3), where nodes $N+1$ and $\ell = 1, ..., N$ represent the master and slaves systems, respectively (*cf.* [8], reference model in the linear case). All systems (1) are said to be in the state of GMS in the sense of next Definition (illustrated in Figure 2).

Definition 4 *GMS of systems (1) is achieved if there exist differential primitive elements that generate the transformations $H_\ell(\cdot) \triangleq (\phi^{-1}_{u_{N+1}} \circ \phi_{u_\ell})(\cdot)$, $H_\ell : \Omega_\ell \to \Omega_{N+1}$; an algebraic synchronization manifold $\mathcal{M}_\ell = \{(x_{N+1}, x_l) \mid x_{N+1} = H_\ell(x_\ell)\}$, a compact set B_ℓ such that $\mathcal{M}_\ell \subset B_\ell \subset \Omega_\ell \times \Omega_{N+1} = D_\ell$ and a dynamical control law that renders this set the stable attractor of the $\ell-$th slave system, such that, $\lim_{t\to\infty} \|x_{N+1} - H_\ell(x_\ell)\|_2 \to 0$ $\forall (x_{N+1}(t_0), x_l(t_0)) \in B_\ell$, $\forall\ \ell = 1, ..., N$.*

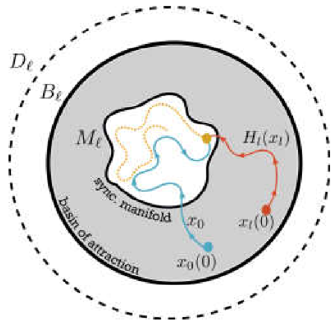


Figure 2: GMS regimen

According to Definition 4, an algebraic synchronization manifold for the entire network can be given by

$$\mathcal{M}_x = \{(x_{N+1} \otimes \mathbf{1}_N, \bar{x}_\ell) \mid H_1(x_1) = ... = H_N(x_N) = x_{N+1}\}. \tag{4}$$

where $\bar{x}_\ell = (x_1^T, ..., x_N^T)^T \in \mathbb{R}^{Nn}$. In what follows its stability is studied. The objective is to impose the dynamics of the master to all slaves in the network such that $\mathcal{M}_x$ is a stable attractor, without loss of generality assume $u_{N+1} = 0$ and that all slaves (3) are coupled by

$$\begin{aligned} \bar{u}_\ell = &- g_\ell(\eta_\ell, U_\ell) + g_{N+1}(\eta_{N+1}) + cE(\eta_{N+1} - \eta_\ell) \\ &- c\sum_{j=1}^{N} a_{\ell j} E\left(\eta_\ell - \eta_j\right) \end{aligned} \tag{5}$$

where c is the coupling strength, $a_{\ell,j}$ are entries of adjacency matrix $\mathcal{A}_{N+1} = \begin{pmatrix} \mathcal{A}_N & \mathbf{1}_N \\ 0_{1\times N} & 0 \end{pmatrix}$. Set matrix $E = \begin{pmatrix} 0_{(n-1)\times n} \\ \mathbf{1}_n^T \end{pmatrix}$, thus closed loop system is given by

$$\dot{\eta} = F(\eta, g_{N+1}(\eta_{N+1})) - c(\mathcal{L}_{N+1} \otimes E)\eta, \tag{6}$$

with $\eta = (\eta_1^T, ..., \eta_{N+1}^T)^T \in \mathbb{R}^{(N+1)n}$, $F(\eta, g_{N+1}(\eta_{N+1})) = (\mathbf{1}_{N+1} \otimes A_n)\eta + \mathbf{1}_{N+1} \otimes B_n g_0(\eta_{N+1})$, $\mathcal{L}_{N+1} = \begin{pmatrix} \mathcal{L}_N + I_N & -\mathbf{1}_N \\ 0_{1\times N} & 0 \end{pmatrix}$, where the Adjacency matrix $\mathcal{A}_N$ and Laplacian matrix $\mathcal{L}_N$ are associated with graph $\mathcal{G}_N = (\mathcal{V}_N, \mathcal{E}_N, \mathcal{A}_N)$ for interactions between slaves systems only. It is clear that if second term in (6) is zero asymptotically, then $\eta_\ell \to \eta_0$ as $t \to \infty$, $\forall \ell = 1, ..., N$. To that end, define vector $\bar{\eta}_\ell \triangleq (\eta_1^T, ..., \eta_N^T)^T \in \mathbb{R}^{Nn}$ and synchronization error $e \triangleq \eta_{N+1} \otimes \mathbf{1}_N - \bar{\eta}_\ell = (e_1, ..., e_N)^T \in \mathbb{R}^{Nn}$, taking the time derivative of e and after some algebraic manipulations $\dot{e} = \Xi e$ where $\Xi = (I_N \otimes (A_n - cE) - c\mathcal{L}_N \otimes E)$, then assume that (**i**) $\exists\, c$ such that Ξ is a Hurwitz matrix.

Now, since all nodes behave asymptotically in the same fashion (in transformed coordinates), algebraic synchronization manifold for closed loop system (6) is given by

$$\mathcal{M}_\eta = \{(\eta_{N+1} \otimes \mathbf{1}_N, \bar{\eta}_\ell) \mid \eta_1 = ... = \eta_N = \eta_{N+1}\}. \tag{7}$$

From straightforward calculations (see [5] for details) the next relation of distances fulfills $\|(\eta_{N+1} \otimes \mathbf{1}_N - \bar{\eta}_\ell)\|_2 = \sqrt{2}\|(\eta_{N+1} \otimes \mathbf{1}_N, \bar{\eta}_\ell)\|_{\mathcal{M}_\eta}$. As Ξ is a Hurwitz matrix we can find $P, Q > 0$ such that $P\Xi + \Xi^T P = -Q$. Let $V = e^T P e$, by using Rayleigh-Ritz inequality, then

$$2\lambda_{min}(P)\|(\eta_{N+1} \otimes \mathbf{1}_N,\, \bar{\eta}_\ell)\|^2_{\mathcal{M}_\eta} \le V \le 2\lambda_{max}(P)\|(\eta_{N+1} \otimes \mathbf{1}_N,\, \bar{\eta}_\ell)\|^2_{\mathcal{M}_\eta},$$
$$\dot{V} = e^T\left(P\Xi + \Xi^T P\right)e = -e^T Q e \le -\lambda_{min}(Q)\|e\|_2^2,$$

It is not hard to see from previous inequalities and comparison Lemma that

$$\|(\eta_{N+1}(t) \otimes \mathbf{1}_N, \bar{\eta}_\ell(t))\|_{\mathcal{M}_\eta} \le \alpha e^{-\frac{\beta}{2}t}\|(\eta_{N+1}(0) \otimes \mathbf{1}_N, \bar{z}_\ell(0))\|_{\mathcal{M}_\eta} \tag{8}$$

with $\alpha \triangleq \sqrt{\lambda_{max}(P)/\lambda_{min}(P)}$ and $\beta \triangleq \lambda_{min}(Q)/\lambda_{max}(P)$.

On the other hand, from Lemma 1 note that $\eta_{N+1} = \phi_{N+1}(x_{N+1})$ and $\bar{\eta}_\ell = (\phi_{u_1}(x_1)^T, ..., \phi_{u_N}(x_N)^T)^T$, assuming (**ii**) $\phi^{-1}_{u_{N+1}}(\cdot)$ is a diffeomorphic (and uniformly bounded) function with Lipschitz constant $L > 0$, then

$$\left\|\mathbf{1}_N \otimes x_{N+1} - \begin{pmatrix} H_1(x_1) \\ \vdots \\ H_N(x_N) \end{pmatrix}\right\|_2^2 = \sum_{\ell=1}^{N} \|\phi^{-1}_{u_{N+1}}(\eta_{N+1}) - \phi^{-1}_{u_{N+1}}(\eta_\ell)\|_2^2$$
$$\le L^2 \sum_{\ell=1}^{N} \|\eta_{N+1} - \eta_\ell\|_2^2.$$

It immediately follows from (7), (8) and above inequality that

$$\left\|\mathbf{1}_N \otimes x_{N+1} - \begin{pmatrix} H_1(x_1) \\ \vdots \\ H_N(x_N) \end{pmatrix}\right\|_2 \le \bar{\alpha} e^{-\frac{\beta}{2}t}\|(\eta_{N+1}(0) \otimes \mathbf{1}_N, \bar{z}_\ell(0))\|_{\mathcal{M}_\eta}$$

with $\bar{\alpha} = \alpha L\sqrt{2}$. Therefore, GMS for systems (1) is achieved, i.e., $H_\ell(x_\ell) \to x_{N+1}$ as $t \to \infty$, $\forall \ell = 1, ..., N$. Finally, next conclusion is established.

Theorem 1 *Consider a network of $N+1$ heterogeneous systems* (1)*, as nodes in the directed spanning tree $\mathcal{G}_{N+1}$, that can be transformed into a network of systems in their GOCF* (3) *coupled by* (5)*. If assumptions (**i**) and (**ii**) are fulfilled, then GMS is achieved, i.e., the algebraic differential manifold $\mathcal{M}_x$ in* (4) *is asymptotically stable.* □

Notice that GMS is achieved for any type of interplay $\mathcal{G}_N$ between of slaves systems as these receive complete information from its master system (see Figure 1).

4. Numerical example

To illustrate Theorem 1, consider the numerical example given in Ref. [5] with a directed spanning tree $\mathcal{G}_3$ with Rössler, Chua and Colpitts systems as nodes (see Figure 3); where $\mathcal{A}_3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathcal{L}_3 = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. Thus,

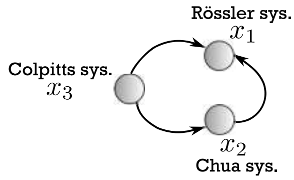


Figure 3: Directed spanning tree in numerical example

individual dynamics for nodes $i = 1, 2$ are given by

$$\begin{aligned} \dot{x}_{11} &= -(x_{12} + x_{13}), \\ \dot{x}_{12} &= x_{11} + a_1 x_{12}, \\ \dot{x}_{13} &= b_1 + x_{13}(x_{11} - c_1), \\ y_1 &= x_{12} + u_1. \end{aligned}$$

and

$$\begin{aligned} \dot{x}_{21} &= a_2\left(x_{22} - x_{21} - v_{x_2}\right), \\ \dot{x}_{22} &= x_{21} - x_{22} + x_{23}, \\ \dot{x}_{23} &= -b_2 x_{22}, \\ v_{x_2} &= m_1 x_{21} + 0.5\,(m_2 - m_1)\,(|\, x_{21} + 1\,| - |\, x_{21} - 1\,|), \\ y_2 &= x_{23} + u_2 \end{aligned}$$

with coordinate transformation given by

$$\phi_{u_1}(x_1) := \begin{pmatrix} \eta_{11} \\ \eta_{12} \\ \eta_{13} \end{pmatrix} = \begin{pmatrix} x_{12} + u_{11} \\ x_{11} + a_1 x_{12} + u_{12} \\ a_1 x_{11} + (a_1^2 - 1)x_{12} - x_{13} + u_{13} \end{pmatrix},$$

and

$$\phi_{u_2}(x_2) := \begin{pmatrix} \eta_{21} \\ \eta_{22} \\ \eta_{23} \end{pmatrix} = \begin{pmatrix} x_{23} + u_{21} \\ -b_2 x_{22} + u_{22} \\ -b_2(x_{21} - x_{22} + x_{23}) + u_{23} \end{pmatrix},$$

respectively. And for node $i = 3$

$$\begin{aligned} \dot{x}_{31} &= -a_3 \exp(-x_{32}) + a_3 x_{33} + a_3, \\ \dot{x}_{32} &= b_3 x_{33}, \\ \dot{x}_{33} &= -c_3 x_{31} - c_3 x_{32} - d_3 x_{33}, \\ y_3 &= x_{32}, \end{aligned}$$

note that $u_3 = 0$ and inverse transformation $\phi^{-1}_{u_3} \in C^1$ is given by

$$\phi^{-1}_{u_3}(\eta_3) := \begin{pmatrix} x_{31} \\ x_{32} \\ x_{33} \end{pmatrix} = \begin{pmatrix} (-\eta_{33} - d_3\eta_{32} - b_3 c_3 \eta_{31})\,/b_3 c_3 \\ \eta_{31} \\ \eta_{32}/b_3 \end{pmatrix},$$

where $\|[\partial/\partial\eta_3]\phi_{u_3}^{-1}(\eta_3)\|_\infty = (b_3c_3 + d_3 + 1)/b_3c_3 =: \bar{L}$, $L \leq \sqrt{3}\bar{L}$. Assume positive constant parameters and initial conditions as in Table 1 and Table 2 ensuring chaotic behaviour. And taking into account dynamical control

Table 1: Parameters for network $\mathcal{G}_3$

i	a_i	b_i	c_i	d_i	m_i
1	0.2	0.2	5	-	$-5/7$
2	15	25.58	-	-	$8/7$
3	6.2723	6.2723	0.0797	0.6898	-

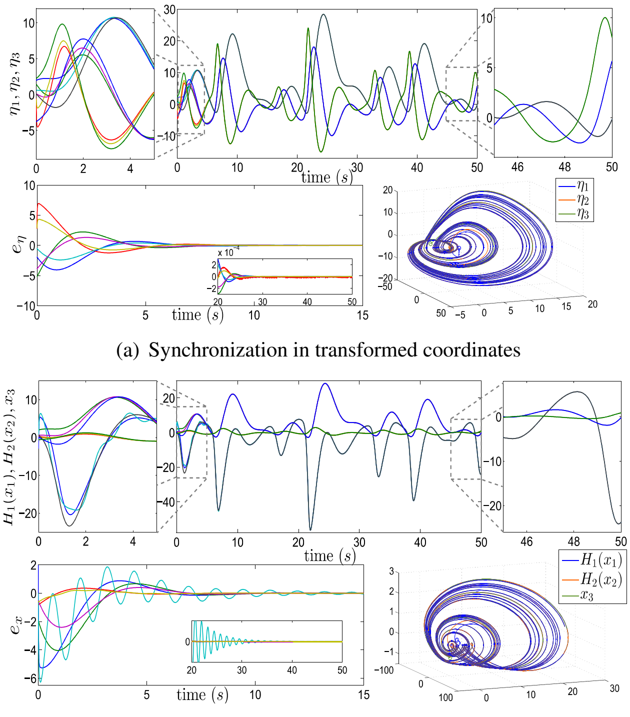
laws (5) with coupling strength $c = 50$, such that conditions of Theorem 1 are fulfilled. Then synchronization in transformed coordinates and GMS are achieved (see Figure 4) where $H_1(x_1(t)) := \phi_{u_3}^{-1} \circ \phi_{u_1}(x_1(t))$ and $H_2(x_2(t)) := \phi_{u_3}^{-1} \circ \phi_{u_2}(x_2(t))$. Finally an asymptotically stable synchronization algebraic manifold for network $\mathcal{G}_3$ is described by:

$$\mathcal{M}_x = \{(x_3 \otimes \mathbf{1}_2, \bar{x}_\ell) \mid x_3 = H_1(x_1) = H_2(x_2)\}$$

with $\bar{x}_\ell = (x_1^T, x_2^T)^T$.

Table 2: Initial conditions

i	$x_i(0)$
1	$(1\ 2\ -5)^T$
2	$(0.6\ 0.1\ 0.6)^T$
3	$(0.6\ 0.1\ -0.6)^T$



(a) Synchronization in transformed coordinates

(b) Generalized multi-synchronization

Figure 4: Synchronization of network $\mathcal{G}_3$ with dynamical control laws (5) and coupling strength $c = 50$; $e_\eta = e$ and e

5. Conclusion and discussion

In a network of distinct interacting chaotic systems, due to differences between their dynamical structure, to find a stable synchronous behavior is a complicated task. Within a differential and algebraic framework, we have presented a general asymptotically stable algebraic synchronization manifold; and from simple algebraic properties of chaotic systems some preliminary results on stability of the synchronization manifold for GMS are given (allowing any type of interplay between slave systems). In particular, an asymptotically stable synchronous behavior can be reached with the design of dynamical control laws, from the knowledge of the differential primitive elements of the network, for all slave systems. It is worth mentioning that in this methodology a stable synchronous behavior is obtained in regard to a single coupling strength c, *cf.* [5]. Forthcoming investigations will include the analysis of GMS under switching interactions between slave nodes.

References

[1] A. Pikovsky, M. Rosenblum, and J. Kurths, Synchronization: A universal concept in nonlinear sciences, Cambridge University Press, 2001.

[2] L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," *Phys. Rev. Lett.*, vol. 64, pp. 821-824, 1990.

[3] A. Isidori, L. Marconi, G. Casadei, "Robust output synchronization of a network of heterogeneous nonlinear agents via nonlinear regulation theory,"", *IEEE Trans. on Autom. Control*, vol. 59, pp. 2680–2691, 2014.

[4] E. Panteley and A. Loría, "Synchronization and dynamic consensus of heterogeneous networked systems," *IEEE trans. on Autom. Control*, vol. 62, pp. 3758–3773, 2017.

[5] C. D. Cruz-Ancona and R. Martnez-Guerra, "Generalized multi-synchronization: A leader-following consensus problem of multi-agent systems," *Neurocomputing*, vol. 233, pp. 52–60, 2017.

[6] L. Kocarev and U. Parlitz, "Generalized synchronization, predictability, and equivalence of unidirectionally coupled dynamical systems," *Phys. Rev. Lett.*, vol. 76, no. 11, pp. 1816–1819, 1996.

[7] R. Martínez-Guerra and C. D. Cruz-Ancona, Algorithms of Estimation for Nonlinear Systems: A Differential and Algebraic Viewpoint, Springer, 2017.

[8] Ren, Wei, and R. W. Beard, Distributed consensus in multi-vehicle cooperative control, Springer, 2008.