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Annihilation of Periodic Orbits in Time Delayed Feedback Controlled Systems

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Abstract—It is well known that the time delayed feedback control has an ability to stabilize unstable periodic orbits targeted in chaotic attractors. On the other hand, the effects of control on non-target periodic orbits have not been discussed. In this paper, harmonic balance analysis of controlled periodic systems demonstrates that non-target orbits annihilate as feedback gain is increased. The annihilation is numerically confirmed in the controlled two-well Duffing system.

1. Introduction

Since the OGY method was proposed [1], stabilizing unstable periodic orbits embedded in chaotic attractors has been extensively studied in the field of nonlinear dynamics [2]. Among several proposed methods, the *time delayed feedback control* [3] has especially occupied the interest of researchers over this decade because of its feasibility to experimental systems. The control method is easily implemented without the exact model of controlled object nor complicated computer processing for reconstruction of underlying dynamics. The target unstable periodic orbits are stabilized by control signal only relying on the difference signal between the present output signal and the past one. The feasibility of the control method has been experimentally demonstrated in a wide variety of systems including electronic circuits [4], laser systems [5], gas charge systems [6], mechanical oscillators [7], chemical systems [8]. In addition, control characteristics has been discussed with focusing on the local stability of target orbits under control input [9]. As one of notable results, the odd number condition has been derived first for discrete systems [10] and subsequently extended to continuous systems [11, 12]. The odd number condition provides a class of unstable periodic orbits that cannot be stabilized by the control method and extended ones [13, 14]. An improved version of the control method has been proposed to circumvent the odd number condition [15].

On the other hand, there still remain open problems on clarification of control performance [9]. In particular, the global dynamics and related control characteristics have not been clarified [16]. The global dynamics of a controlled system is described by the structure of an infinite dimensional phase space, which is function space due to the time delay including feedback loop. So far, the authors have discussed the global dynamics in function space for the controlled two-well Duffing system. They reported that the original chaos producing structure is simplified under large feedback gain and then control performance is improved [18]. The observed change from irregular to smooth basin boundary [17] and quick convergence to targets without long chaotic transient [18] suggest that the perturbation by control signal changes not only the stability of the target orbits but also the global structure of phase space. However, there has been no explanation on the mechanism changing global phase structure. The clarification of the mechanism is quite important in practice of controlling chaos, because the chaotic attractors are possibly destroyed by the stability change of target orbits. The destruction of chaotic attractors causes long chaotic transient and complicated basin structure against the purpose of control.

In this paper, we discuss the effects of control signal on non-target periodic orbits in periodic system under time delayed feedback control. We particularly focus on the annihilation of the non-target orbits whose periods are integer multiples of the time delay for the increase of feedback gain. The harmonic balance analysis [21] of the periodic systems under control demonstrates that the non-target orbits are destroyed under large feedback gain, while the existence and location of the periodic orbits with the target period are not influenced by control inputs. The annihilation of non-target orbits are closely related to the global phase structure of the controlled systems [19]. Since the non-target orbits are originally embedded in chaotic attractor, it is expected that the original global structure producing chaos is simplified after the annihilation of non-target orbits. The suggested annihilation is confirmed for the two-well Duffing system under control using harmonic balance method.

2. Harmonic balance analysis of periodic system under control

As first suggested by Pyragas [3], unstable periodic orbits embedded in chaotic attractors can be stabilized by using continuous feedback of the present output signals and the past ones. In particular, control of unstable period-τ orbits in a chaotic system is described as follows:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{F}(t,\, \boldsymbol{x},\, \boldsymbol{u}) \\ \boldsymbol{u} = K[\boldsymbol{g}(\boldsymbol{x}_\tau) - \boldsymbol{g}(\boldsymbol{x})], \end{cases} \tag{1}$$

where $\dot{\boldsymbol{x}} = \boldsymbol{F}(t,\, \boldsymbol{x},\, \boldsymbol{u})$ corresponds to the chaotic system, $\boldsymbol{x} = \boldsymbol{x}(t)$ and $\boldsymbol{x}_\tau = \boldsymbol{x}(t-\tau)$ denote the current and past state vector of the system, respectively, and $\boldsymbol{u}$ shows control signal. The uncontrolled system $\dot{\boldsymbol{x}} = \boldsymbol{F}(t,\, \boldsymbol{x},\, \boldsymbol{0})$

generates a chaotic attractor which contains the target unstable period-τ orbits. The control method is easily implemented to experimental systems without identifying the exact model of the chaotic system and reconstructing underlying dynamics from experimental data. The control signal $\boldsymbol{u}(t)$ is simply obtained by the product of feedback gain K and the difference between the present output signal $\boldsymbol{g}(\boldsymbol{x})$ and the past output $\boldsymbol{g}(\boldsymbol{x}_\tau)$, as shown in Eq. (1). τ is the time delay adjusted to the period of the target unstable periodic orbit embedded in the chaotic attractor. If the time delay is precisely adjusted, convergence of solutions to the target orbits makes the controlled system (1) degenerate from an infinite dimensional system with time delay to the original finite dimensional one. It follows that the control signal converges to null when control is achieved. Thus, both number and location of the orbits with the same period as the targets are conserved under any feedback gain [20].

On the other hand, those of non-target orbits are changed in general. In the following, we demonstrate the orbits whose periods are integer multiples of the time delay are expected to annihilate as feedback gain is increased. The harmonic balance method is applied to approximate the steady state solutions in periodic systems under time delayed feedback control. The harmonic balance method is a well known approach to analyze nonlinear oscillation in physical systems [21]. A periodic solution of a nonlinear system is represented as a truncated Fourier series expansion with unknown Fourier coefficients. The approximated solution is obtained by determining the coefficients as a solution of nonlinear algebraic equations that derived by substituting the Fourier series to the differential equation.

We here consider the following n-dimensional periodic system under time delayed feedback control:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(t,\, \boldsymbol{x}) + \boldsymbol{u}, \\ \boldsymbol{u} = K[\boldsymbol{x}_\tau - \boldsymbol{x}], \end{cases} \tag{2}$$

where $\boldsymbol{f}(t+T,\, \boldsymbol{x}) = \boldsymbol{f}(t,\, \boldsymbol{x})\,;\, \omega = 2\pi/T$ denotes the periodic system and hereafter frequency component of ω is called *fundamental* component. The time delay τ is here adjusted to the fundamental period T of $\boldsymbol{f}$. K shows n-dimensional feedback gain matrix. The above equation is a special form of Eq. (1) obtained under $\boldsymbol{F}(t,\, \boldsymbol{x},\, \boldsymbol{u}) = f(t,\, \boldsymbol{x}) + u$ and $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}$. For the further discussion, Eq. (2) is rewritten as follows:

$$\dot{\boldsymbol{x}} - \boldsymbol{f}(t,\, \boldsymbol{x}) = K[\boldsymbol{x}_\tau - \boldsymbol{x}]. \tag{3}$$

Suppose that the system (2) has a period-mT orbit $\boldsymbol{x}(t)$ satisfying $\boldsymbol{x}(t) = \boldsymbol{x}(t + mT)(m \geq 1; \text{integer})$ for any t. Then $\boldsymbol{x}(t)$ is approximated by the following truncated Fourier series expansion:

$$\boldsymbol{x}(t) = \boldsymbol{a}_0 + \sum_{k=1}^{N} \left(\boldsymbol{a}_{2k-1} \cos \frac{k}{m}\omega t + \boldsymbol{a}_{2k} \sin \frac{k}{m}\omega t \right), \tag{4}$$

where $\boldsymbol{a}_i = (a_{i1},\, a_{i2},\, \cdots,\, a_{in})^T \in \mathbb{R}^n (i = 0,\, 1,\, 2,\, \cdots,\, 2N)$ denotes Fourier coefficient vector. N is the maximum order of component which should be taken into account to obtain approximate solutions. From Eq. (4), the $\dot{\boldsymbol{x}}(t)$ and $\boldsymbol{x}(t-\tau) - \boldsymbol{x}(t)$ are obtained as follows:

$$\dot{\boldsymbol{x}}(t) = \frac{\omega}{m} \sum_{k=1}^{N} \left(k\boldsymbol{a}_{2k} \cos \frac{k}{m}\omega t - k\boldsymbol{a}_{2k-1} \sin \frac{k}{m}\omega t \right), \tag{5}$$

$$\boldsymbol{x}(t-\tau) - \boldsymbol{x}(t) \tag{6}$$

$$= \sum_{k=1}^{N} \Bigg[\left\{ \boldsymbol{a}_{2k-1} \left(\cos \frac{k}{m}\omega\tau - 1 \right) - \boldsymbol{a}_{2k} \sin \frac{k}{m}\omega\tau \right\} \times \cos \frac{k}{m}\omega t + \left\{ \boldsymbol{a}_{2k-1} \sin \frac{k}{m}\omega\tau + \boldsymbol{a}_{2k} \left(\cos \frac{k}{m}\omega\tau - 1 \right) \right\} \times \sin \frac{k}{m}\omega t \Bigg]. \tag{7}$$

Substituting Eqs. (4) (5) and (7) into Eq. (3) and equating every component, one can derive the following equations determining the Fourier coefficient vectors:

$$\boldsymbol{\mathcal{F}}_i^m(a) = K\boldsymbol{\mathcal{U}}_i^m(a) \quad (i = 0,\, 1,\, \cdots,\, 2N), \tag{8}$$

where $a = (\boldsymbol{a}_0^T,\, \boldsymbol{a}_1^T,\, \cdots,\, \boldsymbol{a}_{2N}^T)^T$. $\boldsymbol{\mathcal{F}}_i^m(a)$ and $\boldsymbol{\mathcal{U}}_i^m(a)$ are defined for $k = 1,\, 2,\, \cdots,\, N$ in the following:

$$\begin{cases} \boldsymbol{\mathcal{F}}_0^m(a) = -\dfrac{1}{mT} \displaystyle\int_0^{mT} \boldsymbol{f}(t,\, \boldsymbol{x}(t)) \mathrm{d}t \\ \boldsymbol{\mathcal{F}}_{2k-1}^m(a) = \dfrac{\omega}{m} k\boldsymbol{a}_{2k} \\ \qquad\qquad -\dfrac{2}{mT} \displaystyle\int_0^{mT} \boldsymbol{f}(t,\, \boldsymbol{x}(t)) \cos \frac{k}{m}\omega t \mathrm{d}t, \\ \boldsymbol{\mathcal{F}}_{2k}^m(a) = -\dfrac{\omega}{m} k\boldsymbol{a}_{2k-1} \\ \qquad\qquad -\dfrac{2}{mT} \displaystyle\int_0^{mT} \boldsymbol{f}(t,\, \boldsymbol{x}(t)) \sin \frac{k}{m}\omega t \mathrm{d}t \end{cases} \tag{9}$$

$$\begin{cases} \boldsymbol{\mathcal{U}}_0^m(a) = 0, \\ \boldsymbol{\mathcal{U}}_{2k-1}^m(a) = \left\{ \boldsymbol{a}_{2k-1} \left(\cos \frac{k}{m}\omega\tau - 1 \right) - \boldsymbol{a}_{2k} \sin \frac{k}{m}\omega\tau \right\}, \\ \boldsymbol{\mathcal{U}}_{2k}^m(a) = \left\{ \boldsymbol{a}_{2k-1} \sin \frac{k}{m}\omega\tau + \boldsymbol{a}_{2k} \left(\cos \frac{k}{m}\omega\tau - 1 \right) \right\}. \end{cases} \tag{10}$$

Note that $\boldsymbol{\mathcal{U}}_{2k-1}^m(a) = \boldsymbol{\mathcal{U}}_{2k}^m(a) = 0$ for $k = jm(j = 0,\, 1,\, \cdots)$, since $\cos \frac{k}{m}\omega\tau - 1 = \sin \frac{k}{m}\omega\tau = 0$. The relations are considered as constraint conditions for solving remaining equations. For $m = 1$, these relations confirm that both the existence and location of orbits with period-τ is independent of feedback gain, since $\boldsymbol{\mathcal{U}}_i^m(a) = 0$ for all i, that is, the control input do not have a direct current component, the fundamental and its higher harmonics.

As for the periodic solutions with $m \geq 2$, one can expect that they cease to exist for large feedback gain, because one can find K for some i such that

$$||\boldsymbol{\mathcal{F}}_i^m(a) - K\boldsymbol{\mathcal{U}}_i^m(a)|| \geq \big| ||\boldsymbol{\mathcal{F}}_i^m(a)|| - ||K\boldsymbol{\mathcal{U}}_i^m(a)|| \big| > 0. \tag{11}$$

in a given bounded domain including the origin and satisfying the constraint conditions, if $\boldsymbol{\mathcal{F}}^m_{2k-1}(0) = \boldsymbol{\mathcal{F}}^m_{2k}(0) = 0$ for $k \neq jm$ and $\boldsymbol{\mathcal{F}}^m_i(a)$ is bounded in the given domain. In particular, if K is a diagonal matrix, that is, $K = \mathrm{diag}(K_1, K_2, \cdots, K_n); K_i \geq 0$, the above inequality is satisfied if one of the elements is sufficient large. If the $\boldsymbol{\mathcal{F}}^m_i(a)$ are polynomial in a, it is also conjectured that the periodic solutions satisfying Eqs. (8) annihilate at a certain feedback gain, because the solutions of Eqs. (8) are given by the intersecting points of the surface and plane built by the left-hand sides and right-hand sides of Eqs. (8) in the Euclidean space $\mathbb{R}^{n(2N+1)}$, respectively. As the planes are sloped by increasing feedback gain, a pair of intersecting points approach each other and finally coincide when the plane are tangent to the surface. We should note that another possibility that non-target orbits degenerate into target ones without annihilation. The degeneration occurs, when the amplitude of the fundamental component increases, while that of subharmonic ones converge to zero, so that the equality of Eqs. (8) remains satisfied for the increase of feedback gain. It seems that the degeneration of non-target orbits is related to the period-doubling bifurcation of target orbits, which is often observed in time delayed feedback controlled systems.

In this section, the annihilation of non-target periodic orbits in the controlled system has been discussed based on the harmonic balance method. We show that non-target periodic orbits with period-mT annihilate as feedback gain is increased. In the next section, the conjecture is numerically confirmed for the controlled two-well Duffing system.

3. Annihilation of non-target orbits in controlled Two-well Duffing System

The two-well Duffing system is a model for the first-mode vibration in the magnetoelastic beam system under sinusoidal forcing [22]. The two-well Duffing system is here controlled by the signal $u(t)$:

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = -\delta y(t) + x(t) - x(t)^3 + A\cos\omega t + u(t) \\ u(t) = K[y(t-\tau) - y(t)], \end{cases} \tag{12}$$

where $x(t)$ and $y(t)$ denote the displacement and velocity of the two-well Duffing system, respectively. The $u(t)$ is generated from the difference signal between the current velocity and past one. The parameter of the original system is here fixed at $(\delta, A, \omega) = (0.3, 0.34, 1.0)$, where the system generates the chaotic attractor [23]. δ denotes the damping coefficient. A represents the forcing amplitude and ω the frequency. The detailed dynamics under $\omega = 1.0$ was summarized in [23]. τ is adjusted to 2π for stabilizing two symmetric inversely unstable periodic orbits with period-2π

We here apply the harmonic balance method to approximate period-6π orbits of the two-well Duffing system by a

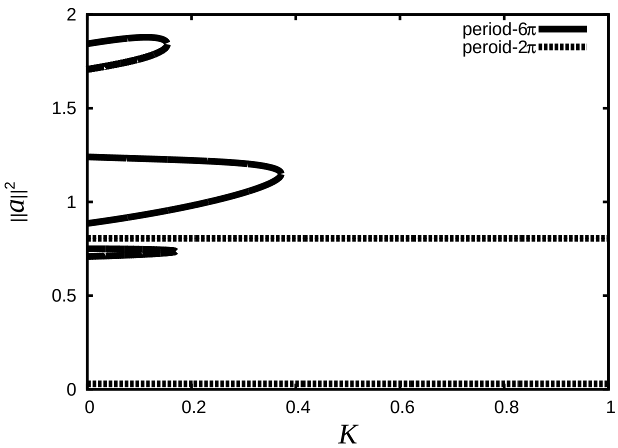

Figure 1: Amplitude characteristic of non-target period-6π orbits (solid) and target period-2π orbits (dashed).

truncated Fourier series as follows:

$$x(t) = A_0 + A_1\cos t + B_1 \sin t + A_2 \cos\frac{1}{3}t + B_2 \sin\frac{1}{3}t. \tag{13}$$

Substituting Eq.(13) into Eq.(12) and equating each component, one can derive the following equations:

$$A_0\left(A_0^2 + \frac{3}{2}A_1^2 + \frac{3}{2}B_1^2 + \frac{3}{2}A_2^2 + \frac{3}{2}B_2^2\right) - A_0 = 0, \tag{14}$$

$$3A_1\left(A_0^2 + \frac{1}{4}A_1^2 + \frac{1}{4}B_1^2 + \frac{1}{2}A_2^2 + \frac{1}{2}B_2^2\right) \tag{15}$$
$$+ \frac{1}{4}B_2(3B_2^2 - A_2^2) - 2A_1 + \delta B_1 - B = 0,$$

$$3B_1\left(A_0^2 + \frac{1}{4}A_1^2 + \frac{1}{4}B_1^2 + \frac{1}{2}A_2^2 + \frac{1}{2}B_2^2\right) \tag{16}$$
$$+ \frac{1}{4}B_2(B_2^2 - 3A_2^2) - 2B_1 - \delta A_1 = 0,$$

$$3A_2\left(A_0^2 + \frac{1}{2}B_1^2 + \frac{1}{2}A_1^2 + \frac{1}{4}B_2^2 + \frac{1}{4}A_2^2\right) - \frac{10}{9}A_2 + \frac{\delta}{3}B_2 \tag{17}$$
$$+ \frac{3}{4}A_1(A_2^2 - B_2^2) + \frac{3}{2}B_1A_2B_2 = \frac{K}{2}\left(-B_2 + \frac{A_2}{\sqrt{3}}\right),$$

$$3B_2\left(A_0^2 + \frac{1}{2}A_1^2 + \frac{1}{2}B_1^2 + \frac{1}{4}A_2^2 + \frac{1}{4}B_2^2\right) - \frac{10}{9}B_2 - \frac{\delta}{3}A_2, \tag{18}$$

$$+ \frac{3}{4}B_1(A_2^2 - B_2^2) - \frac{3}{2}A_1A_2B_2 = \frac{K}{2}\left(\frac{B_2}{\sqrt{3}} + A_2\right). \tag{19}$$

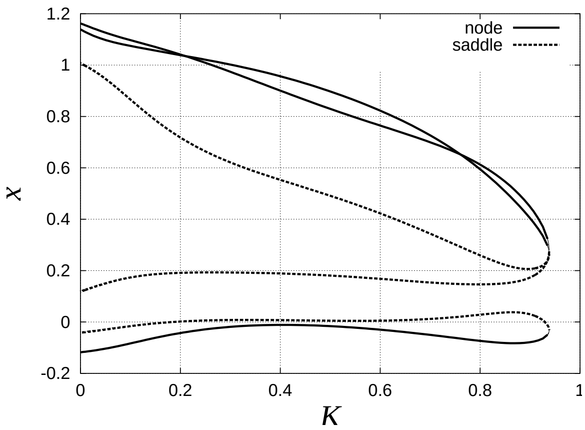

Figure 2: Period-6π orbits numerically traced for feedback gain [19]. The orbits originally embedded in chaotic attractor annihilate by saddle-node bifurcation for large feedback gain.

Figure 1 shows amplitude characteristics of the period-2π orbits (target) and period-6π orbits (not target) obtained by numerically solving Eq. (14)-Eq. (19). Each solid curve shows an amplitude characteristic of a period-6π orbit. We observe six period-6π curves having roots in $K = 0$ and then confirm that each curve annihilates with the corresponding one as feedback gain is increased. On the other hand, the orbits with period-2π denoted by the dashed line do not change their location for any feedback gain. The result here obtained qualitatively agrees with the our previous result showing that non-target period-6π orbits annihilate by saddle-node bifurcation as shown in Fig. 2 [19].

4. Concluding Remarks

In this paper, we have discussed the effect of time delayed feedback control on non-target unstable periodic orbits embedded in chaotic attractors. By applying the harmonic balance method to controlled periodic systems, we demonstrated that the increase of feedback gain causes the annihilation of the non-target orbits whose periods are integer multiples of time delay. The annihilation of the non-target orbits originally embedded in chaotic attractor suggests that the global phase structure under control become simple as feedback gain is increased.

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References

[1] E. Ott, C. Grebogi and A. Yorke, *Phys. Rev. Lett.*, vol.64, pp.1196–1199, 1990.

[2] H. G. Schuster, ed., “Handbook of Chaos Control,”(Wiley-VCh, 1999).

[3] K. Pyragas, *Phys. Lett. A*, vol. 170, pp.421–428, 1992.

[4] K. Pyragas and A. Tamaševičius, *Phys. Lett. A*, vol.204, pp.255–262, 1993.

[5] S. Bielawski, D. Derozier and P. Glorieux, *Phys. Rev. E*, vol.49, pp.R971–R974, 1994.

[6] Th. Pierre, G. Bonhomme and A. Atipo, *Phys. Lev. Rett*, vol.76, pp.2290–2293, 1996.

[7] T. Hikihara and T. Kawagoshi, *Phys. Lett. A*, vol.211, pp.29–36, 1996.

[8] P. Parmananda, R. Madrigal, M. Rivera, L. Nyikos, I. V. Kiss and V. Gáspár, *Phys. Rev. E*, vol.59, pp.5266–5271, 1999.

[9] W. Just, E. Reibold, K. Kacperski, P. Fronczak, J. A. Holyst and H. Benner, *Phys. Rev. E*, vol.61, pp.5045–5056, 2000.

[10] T. Ushio, *IEEE Trans. CAS-I*, vol.43, pp.815–816, 1996.

[11] H. Nakajima and Y. Ueda, *Physica D*, vol.111, pp.143–150, 1998.

[12] W. Just, T. Bernard, M. Ostheimer, E. Reibold and H. Benner, *Phys. Rev. Lett.*, vol.78, pp.203–206, 1997.

[13] J. E. S. Socolar, D. W. Sukow and D. J. Gauthier, *Phys. Rev. E*, vol.50, pp.3245–3248 , 1994.

[14] K. Pyragas, *Phys. Lett. A*, vol.206, pp.323–330, 1995.

[15] K. Pyragas, *Phys. Rev. Lett.*, vol.86, pp.2265–2268, 2001.

[16] T. Hikihara and Y. Ueda, *CHAOS*, vol.9, pp. 887–889, 1999.

[17] K. Yamasue and T. Hikihara, *Phys. Rev. E*, vol.69, 056209, 2004.

[18] K. Yamasue and T. Hikihara, *in XXI International Congress of Theoretical and Applied Mechanics*, Warsaw, Poland, August 15-21, 2004.

[19] K. Yamasue and T. Hikihara, *in 2004 International Symposium on Nonlinear Theory and its Applications*, Fukuoka, Japan, Nov. 29 - Dec. 3 (2004).

[20] H. Nakajima, *Phys. Lett. A*, vol. 232, pp.207–210, 1997.

[21] C. Hayashi, “Nonlinear Oscillations in Physical Systems,” (McGraw-Hill, Inc., 1964).

[22] F. C. Moon and P. J. Holmes, *Journal of Sound and Vibration*, vol.65, pp.275–296, 1979.

[23] Y. Ueda, H. Nakajima, T. Hikihara and H. B. Stewart, *in Dynamical Systems Approaches to Nonlinear Problems in Systems and Circuits* (Eds. by Fathi M. A. Salam and Mark L. Levi), pp.128–137 (SIAM, Philadelphia, 1988).