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Noise-assisted information transmission: communication channel approach

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Abstract—A threshold system is a typical system that shows noise-induced response to weak input, called stochastic resonance. When this system is considered as a digital communication channel, noise-assisted information transmission can in some cases be observed in the sense that a subthreshold binary signal can be transmitted by adding moderate noise intensity to the system. We elucidate this noise-assisted transmission by the following two steps: we show how transition probabilities are parametrized by noise intensity and then we show how mutual information between input and output signal depends on the transition probabilities. From our analysis, we can show that the appearance of noise-assisted transmission depends on the parametric path in the space of transition probabilities.

1. Introduction

Noise-induced effects in nonlinear systems have recently received considerable attention. In particular, stochastic resonance (SR) [1] has been studied in various systems. SR means that the resonance response of a noisy nonlinear system to a subthreshold signal can be optimized by noise intensity. The basic SR mechanism is often explained using a system model with a bistable potential modulated by a subthreshold sinusoid plus some amount of noise. The subthreshold sinusoid alone cannot overcome the potential barrier of the bistable system, but the addition of noise assists the switching movement between the wells of the bistable potential. This switching timing is occasionally synchronized with the external periodic signal. The power spectrum of the timeseries of the system's state has peaks at the drive-signal frequency and its harmonics. To evaluate the resonance response, we can calculate the SNR from the power spectrum. We use the peak power at the signal period as the signal power and the average power around the signal period as the noise power.

Information theoretical approaches have been used to study SR of aperiodic signals, for continuous or binary signals [2-11]. In these studies, correlation, bit error probability or mutual information is used to measure transmission between input and output. SR in a threshold system with binary signal as input and continuous noise is related to the signal detection problem [7, 8, 10] in previous engineering studies [12]. Suboptimal thresholds can results in noise-assisted detection [13]. When we regard these systems as communication channels, we can find that the SR in such systems is equivalent to noise-assisted signal transmission [9].

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In this paper, we explain noise-assisted information transmission in threshold systems. We resolve this noise-assisted transmission to the following two steps. We show at first how transition probabilities are parametrized by noise intensity and then we show how mutual information between input and output signal depends on the transition probabilities. Therefore, we can show that the appearance of noise-assisted transmission depends on the parametric path in the space of transition probabilities.

2. Noise-assisted transmission in threshold systems

A threshold system is a typical system exhibiting SR. Classification by threshold systems is equivalent to the task in signal detection [12] of deciding whether there is a dc signal in noise. As in Ref. [9, 10], we consider the following signal detection problem. A binary input signal has values 0 (input 0) and 1 (input 1) and prior probabilities for each input bit are defined as p_0 and $p_1 (= 1 - p_0)$, respectively. Noise is Gaussian with mean 0 and variance σ^2. We define $P_0(x)$ and $P_1(x)$ as two probability distributions corresponding to the inputs 0 and 1 with added noise. Figure 1 shows an example of the probability distributions, assuming for simplicity that $p_0 = p_1 = 1/2$. We consider the threshold detection of the signal in the presence of noise, with thresholds as shown in Fig. 1.

Standard signal detection techniques assume complete knowledge concerning the values (levels, ampli-

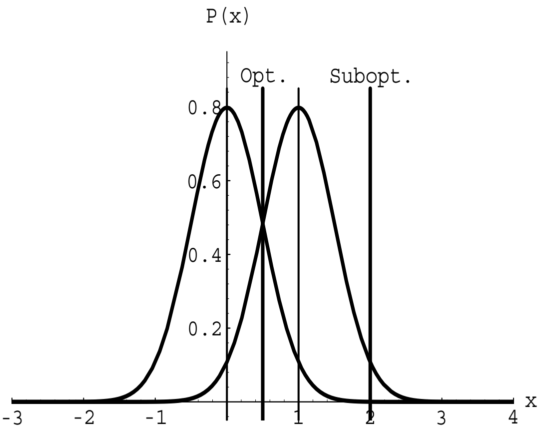

Figure 1: Detection of {0,1} signals with noise intensity $\sigma = 1/2$. Vertical thick lines show examples of thresholds for detection, optimal threshold $\theta_{\text{opt}} = 1/2$ and a suboptimal threshold $\theta = 2$.

tudes) of the binary signals and so we can calculate the optimal threshold θ_{opt}. The optimal threshold maximizes correct detection probability and simultaneously minimizes error probability for decision. All other thresholds are suboptimal. When the values of the binary signals are unknown, we cannot determine the optimal threshold. If we fix the threshold arbitrarily, then a finite noise intensity σ can maximize the detection probability. This is an example of noise-assisted detection. Noise-assisted detection does not occur for all suboptimal thresholds. In this example, suboptimal thresholds located on a value larger than the true value of input 1 or smaller than the value of input 0 show noise-assisted detection. Here we note that the detection probability for suboptimal threshold is always smaller than the one for optimal threshold.

Here we consider information transmission instead of the signal detection. We can regard this threshold system as a communication channel as shown in Fig. 2, characterized by transition probabilities in this channel. Here we define p_{00} as probability that input 0 is

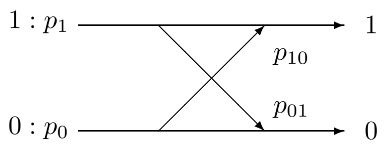

Figure 2: Communication channel model.

detected correctly and also define p_{10} as probability that input 0 is detected as input 1. p_{01} and p_{11} for input 1 are defined similarly. These transition probabilities can be calculated given threshold θ as follows.

$$\begin{aligned} p_{00} &= \int_{-\infty}^{\theta} P_0(x)dx = 1 - p_{10}, \\ p_{10} &= \int_{\theta}^{\infty} P_0(x)dx, \\ p_{01} &= \int_{-\infty}^{\theta} P_1(x)dx, \\ p_{11} &= \int_{\theta}^{\infty} P_1(x)dx = 1 - p_{01}. \end{aligned} \tag{1}$$

We note that p_{00}, p_{10}, p_{01}, and p_{11} are functions of σ, such as $p_{00}(\sigma)$, because $P_0(x)$ and $P_1(x)$ have Gaussian distributions with various σ. We consider p_{10} and p_{01} as independent variables.

To measure signal transmission against noise intensity σ, we can derive input-output mutual information as follows.

$$\begin{aligned} I(\sigma) &= \sum_{\text{in,out}\in\{0,1\}} P(\text{in, out}) \log \frac{P(\text{in, out})}{P_{\text{in}}(\text{in})P_{\text{out}}(\text{out})} \\ &= I(p_0, p_1, p_{10}, p_{01}) \\ &= -\{(1-p_{10})p_0 + p_{01}p_1\} \\ &\quad \times \log\{(1-p_{10})p_0 + p_{01}p_1\} \\ &\quad - \{p_{10}p_0 + (1-p_{01})p_1\} \\ &\quad \times \log\{p_{10}p_0 + (1-p_{01})p_1\} \\ &\quad - p_0 \log p_0 - p_1 \log p_1 \\ &\quad + (1-p_{10})p_0 \log\{(1-p_{10})p_0\} \\ &\quad + p_{10}p_0 \log(p_{10}p_0) + p_{01}p_1 \log(p_{01}p_1) \\ &\quad + (1-p_{01})p_1 \log\{(1-p_{01})p_1\}. \end{aligned} \tag{2}$$

We note that the measure is a function of σ. For suboptimal example ($\theta = 2$), we show mutual information dependence on σ in Fig. 3. We can see the existence of a maximum in the mutual information at non-zero σ. Similar to the previous signal detection case, when the suboptimal threshold is located on a value larger than the value of input 1 or smaller than the value of input 0, we can see noise-assisted information transmission. When the threshold is optimal, information decreases monotonically with increase of σ.

Next we show how the mutual information for a general communication channel is dependent on prior signal probabilities and transition probabilities as in Fig. 2. Figure 4 shows mutual information $I(p_0, p_1, p_{10}, p_{01})$ when p_0 and p_1 are regarded as parameters ($p_0 = p_1 = 1/2$) and p_{10} and p_{01} are regarded as variables.

Since p_{10} and p_{01} are functions of σ, increases of σ draw a path of transition probabilities on p_{10}-p_{01} plane such as in Fig 5. Detection by the optimal threshold ($\theta_{\text{opt}} = 1/2$) corresponds to the path from $(0, 0)$ to

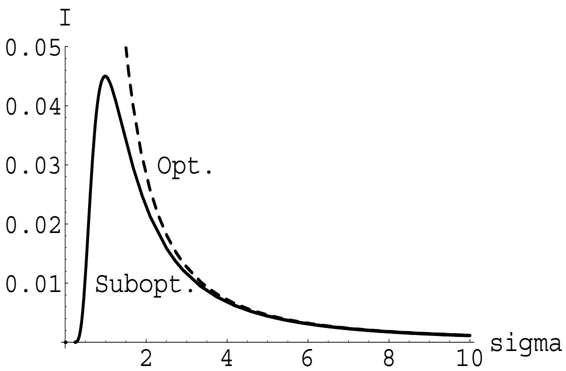

Figure 3: Mutual information I as a function of noise intensity σ. Line shows I using a suboptimal threshold ($\theta = 2$) and dotted line shows I using the optimal threshold ($\theta_{\text{opt}} = 1/2$)

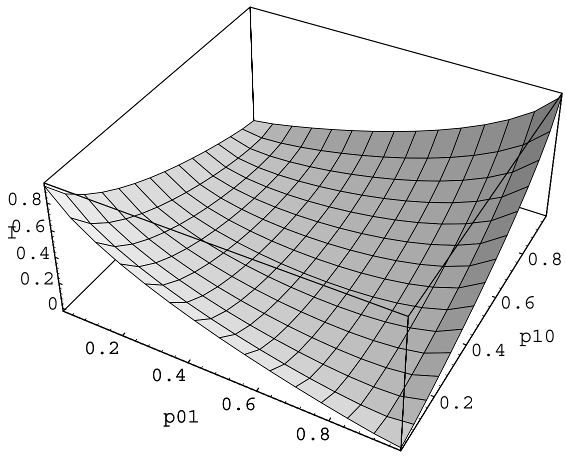

Figure 4: Mutual information I as a function of transition probabilities p_{10} and p_{01}.

$(1/2, 1/2)$. Detection by a suboptimal threshold corresponds to the path from $(1, 0)$ close to $(1/2, 1/2)$ with larger threshold than input 1 value, or from $(0, 1)$ close to $(1/2, 1/2)$ with smaller than input 0 value. These paths are independent of p_0 and p_1.

From the above descriptions, we can determine whether there is noise-assisted transmission or not by following two steps. First, we draw a parametric path for the increasing parameter σ on p_{10}-p_{01} plane as in Fig. 5. Second, the variation of information I with increase of σ can be seen from the variation of I along the path in Fig. 4. When the variation of information is nonmonotonic on the path on p_{10}-p_{01} plane, we can see noise-assisted transmission. The path by the optimal threshold results in decreasing information with increases of σ. By contrast, not all paths for suboptimal thresholds result in nonmonotonic information. When suboptimal threshold is larger than input 1 or smaller than input 0, the paths cause nonmonotonic information and occur noise-assisted transmission.

Here we note that the threshold value changes the path on p_{10}-p_{01} plane. We also note that I surface on p_{10}-p_{01} plane changes depending on p_0, p_1. We emphasize that information curve for suboptimal threshold is always lower than the curve for optimal threshold because of the data processing lemma [14].

3. Conclusions and discussions

We explained noise-assisted information transmission in a threshold system. We resolved this noise-assisted transmission to the following two steps. First, we show how transition probability p_{10} and p_{01} are commonly parametrized by noise intensity σ. This means that increases of σ vary p_{10} and p_{01} and draw a path on p_{10}-p_{01} plane. Second, information I varies as a function of σ along the path. From these considerations, we can find that the path on p_{10}-p_{01} plane determines nonmonotonicity of input-output information with increases of σ. Previous studies have shown dependence of mutual information on noise intensity σ. From our analysis using noise-parametrized transition probabilities, we can determine whether or not there is noise-assisted transmission from the parametric path.

Here we have considered binary signal case, however, we can easily extend our explanation to the case for M kinds of signal values. We have $M(M-1)$ transition probabilities that give a path on $M(M-1)$ dimensional space with increases of σ. Information is calculated through the path depending on increases of σ. We can also easily imagine that it can be extended to continuous signal.

Standard detection assumes that all parameters for signal and noise are known. In general, detection with a lack of prior knowledge may result in noise-assisted detection and information transmission for a arbitrary threshold when the optimal threshold cannot be obtained. Hence, we can utilize nonmonotonic input-output transmission as a non-optimality check for a detector.

Threshold system was considered as a example, however, bistable systems and excitable systems also show noise-assisted transmission. As in Ref. [4], when we use master equations or residence time distributions, we can calculate noise-parametrized transition probabilities in bistable systems to draw a path on p_{10}-p_{01} plane. Detail explanation about bistable systems will be reported elsewhere. We can deal with the excitable systems, similar to bistable systems. First, the two rate distributions with input 0 and with input 1

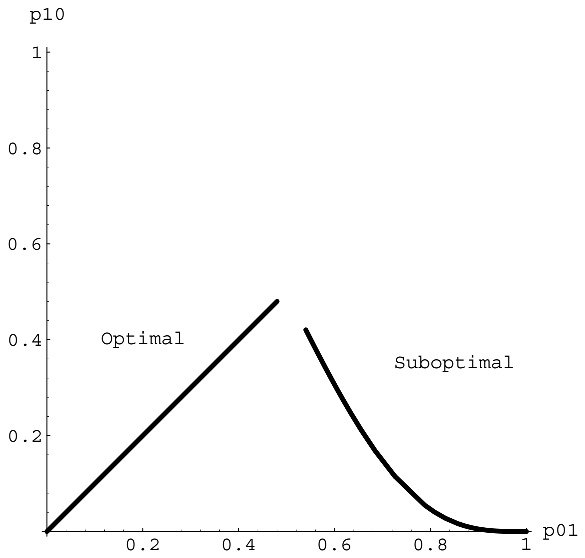

Figure 5: Transition probabilities p_{10} and p_{01} parametrized by σ. A path is drawn for increasing noise intensity σ. One is the case of optimal threshold ($\theta_{\rm opt} = 1/2$), and the other is the case of suboptimal threshold ($\theta = 2$) that results in noise-assisted transmission.

are calculated at each σ by a Fokker-Planck equation using Ref. [15] and so on. Then, transition probability is calculated from the two rate distributions and a path is obtained with increases of σ. We can calculate information dependent on σ and also on the path of transition probabilities.

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