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Generalized Brownian Motion in Systems Forced by Shot Noise

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Abstract–The statistical properties of a generalized diffusion process in which the white noise forcing is replaced by a deterministic or random impulse process are analyzed. General expressions linking the power spectrum and the probability density function of the response to the corresponding properties of the forcing are obtained. It is shown that under certain conditions the probability measure converges to a Wiener process.

1. Introduction

Recently, Nicolis and Nicolis (2003) [1] studied the following differential equation

$$\frac{d\eta}{dt} = \xi(t) \tag{1}$$

and the Langevin-type equation

$$\frac{d\eta}{dt} = -\lambda\eta + \xi(t) \tag{2}$$

with a forcing signal $\xi(t)$ as an output variable of a low-dimensional ergodic deterministic dynamical system. In particular for the signal $\xi(t)$ of the Lorenz system a universal mechanism of deterministic diffusion and a fast convergence of the distribution of the solution $\eta(t)$ to the Gaussian law has been demonstrated for equation (1). Contrary to this case the distribution of $\eta(t)$ in (2) is not attracted to the Gaussian one as the Lindeberg condition is not satisfied [1]. Shimizu in [2] gives an alternative analysis of the Langevin-type equation driven by a deterministic sequence of iterates of a chaotic map.

In this work we focus on equation (1) and demonstrate that deterministic diffusion extends to a large class of dynamical systems forced by a random or chaotic shot noise process $\xi(t)$. The analysis will be carried out using the theory of point processes [3].

2. Random and Deterministic Models of Shot Noise

Shot noise is associated with random fluctuations of current in electrical conductors or electronic devices, due to the fact that the current is carried by discrete charges (electrons). It is important in electronics, telecommunication, and for fundamental physics. A stochastic model for the random shot noise process $\xi(t)$ may be constructed as follows[4]. Suppose that electrons emitted at times $\tau_1, \tau_2, \ldots, \tau_k$ have impulses of magnitudes $A_1, A_2, \ldots, A_k$. Let $N(t)$ be the total number of electrons emitted up to time t

$$N(t) = \max\{k : \tau_k \le t\},$$

then the total signal $\xi(t)$ up to time t can be represented by the model

$$\xi(t) = \sum_{k=1}^{N(t)} A_k \delta\left(t - \tau_k\right) \tag{3}$$

where δ is the Dirac Delta function. Fig. 1 shows a realization of an impulse process.

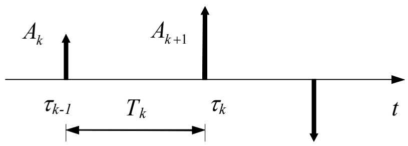

Fig. 1. Schematic visualization of an impulse process

Without loss of generality we assume for the occurrence times τ_k

$$\tau_0 = 0 \ , \tau_k = \tau_{k-1} + T_k = \sum_{i=1}^{k} T_i \ .$$

Random and chaotic shot noise processes will be distinguished according to whether the inter-arrival times T_k are random or deterministic variables, respectively.

A chaotic model for the shot noise $\xi(t)$ has been first introduced in [5]. In this case the magnitudes A_k and inter-arrival times T_k are supposed to be generated by two different chaotic maps:

$$A_{k+1} = \Phi\left(A_k\right), \tag{4}$$

$$T_{k+1} = \varphi\left(T_k\right), \tag{5}$$

characterized by invariant probability measures with the densities $p_A\left(x\right)$ and $p_T\left(y\right)$, respectively.

3. Response to a Shot Noise. The generalized Wiener process

Integrating Eq. (1) we get

$$\eta(t) = \int_0^t \xi(\tau)d\tau = \sum_{k=1}^{\infty} \eta_k rect\left(\frac{t-\tau_{k-1}}{T_k}\right) \tag{6}$$

where $rect(x)=\begin{cases}1, & 0<x<1,\\ 0, & otherwise\end{cases}$. In terms of a counting process $N(t)$ the process $\eta(t)$ can be rewritten in a simple form:

$$\eta(t)=\eta_{N(t)}\,.$$

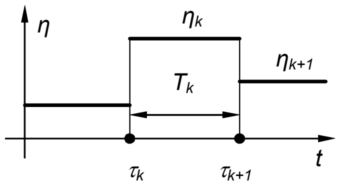

Fig. 2 A solution of Eq. (1)

Thus the process $\eta(t)$ (Fig. 2) is a random/chaotic rectangular signal with step heights η_k satisfying the relation:

$$\eta_k=\eta_{k-1}+A_k \qquad (7)$$

The probability density function (pdf) and power spectral density (psd) for the process $\eta(t)$ will now be calculated.

3.1. Statistical Analysis

3.1.1. Probability density function

We first establish that the distribution function of $\eta(t)$ is

$$P\left(\eta(t)\le z\right)=\sum_{k=1}^{\infty}P_k(t)P\left(\eta_k\le z\right),$$

where $P_k(t)=P\left(\tau_{k-1}<t\le\tau_{k-1}\right)=P\left(N(t)=k\right)$. On the basis of the elementary renewal theorem and $\lim_{t\to\infty}\lim_{k\to\infty}P_k(t)=1$ we asymptotically establish that the pdf of the $\eta(t)$ is the density of the step heights η_k. At the same time from (7) these heights are

$$\eta_k=\eta_{k-1}+A_k=\eta_{k-2}+A_{k-1}+A_k=...=\sum_{i=1}^{k}A_i,\ \left(\eta_0=0\right)$$

Without loss of generality we assume zero mean for the magnitudes A_k

$$\overline{\eta}_k=E\left(\eta_k\right)=k\,\overline{A}=0 \qquad (8)$$

and calculate the variance

$$\sigma_{\eta_k}^2=E\left(\eta_k^{\ 2}\right)-\overline{\eta}_k^{\ 2}=k\sigma_A^2+2\sum_{n=1}^{k-1}(k-n)c_A(n) \qquad (9)$$

where $c_A(k)=E\left(A_iA_{i+k}\right)$ is the autocorrelation function (acf) and $\sigma_A^{\ 2}$ the variance of the A_k.

Hence, for i.i.d. random or uncorrelated chaotic magnitudes A_k ($c_A(k)$=0) we have $\sigma_{\eta_k}^2=k\sigma_A^2$. For correlated random/chaotic A_k we assume that the first moment of the autocorrelation function $c_A(k)$ is finite

$$\sum_{k=1}^{\infty}k\left|c_A(k)\right|<\infty \qquad (10)$$

and then the variance is given by

$$\sigma_{\eta_k}^2=k\sigma_A^2+o(1)\,. \qquad (11)$$

This property can be easily illustrated with the exponentially decaying acf

$$c_A(k)=\sigma_A^2\cdot r^{|k|},\ \ |r|<1. \qquad (12)$$

Note that the chaotic magnitudes A_k generated by the piecewise linear onto maps (12) [6]. We next substitute the acf (12) into (9) and get

$$\sigma_{\eta_k}^2=k\sigma_A^2\frac{1-r}{1+r}-\sigma_A^2\frac{2r\left(1-r^k\right)}{\left(1-r\right)^2}$$

This confirms (11) at large k as $r^k\to 0$.

The acf of the non-stationary sequence of step heights $c_\eta(k,n)=E\left(\eta_k\eta_n\right)$ is given by

$$c_\eta(k,n)=\begin{cases}k\sigma_A^2+\sum_{j=1}^{k-1}\left((2k-j)c_A(j)+(k-j)c_A(n+j)\right)\\ +k\sum_{j=0}^{n-k}c_A(k+j),\ k<n;\end{cases}$$

Let us introduce a new variable $\zeta_k=\frac{\eta_k-\overline{\eta}_k}{\sigma_{\eta_k}}$ with $E\left(\zeta_k\right)=0$ and $E\left(\zeta_k^{\ 2}\right)=1$. It can be shown that ζ_k converges in distribution to the standard normal law, i.e. the central limit theorem holds both with i.i.d.random [7] and chaotic magnitudes A_k [8]. In [3], authors have presented the analytical expressions for the characteristic functions of the chaotic partial sums η_k of the magnitudes A_k generated by PWL onto maps and shown their fast convergence to the limit $\exp\left(-\omega^2/2\right)$.

A weak invariance principle often accompanies the central limit theorem in the theory of random processes and in that of chaotic dynamical systems. We consider a piecewise constant function $W_k(t)$ on $t\in[0,1]$ such that

$$W_k(t)=\frac{\eta_{\lfloor kt\rfloor}}{\sigma\sqrt{k}}\,, \qquad (13)$$

where $\lfloor x\rfloor$ is the floor function (it gives the greatest integer less than or equal to x)

$$\sigma^2=c_A(0)+2\sum_{n=1}^{k-1}c_A(n)\,. \qquad (14)$$

Then for any k $\{W_k\}$ induces a measure on the space of continuous functions on [0,1]. According to the invariance principle this measure converges weakly, as $k\to\infty$, to the Wiener process W (Donsker theorem). Fig. 3 depicts examples of functions $\{W_k\}$ for different k when the magnitudes A_k are chaotic variables generated by a tent map on [-1,1]:

$$A_{n+1}=1-2\left|A_n\right|,n=1,2,...$$

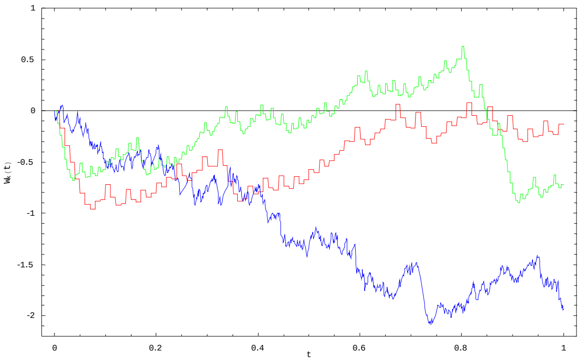

Fig. 3 Three realizations of the process W for k=100,300 and 10000 (red, green and blue line)

The weak invariance principle known also as the functional central limit theorem provides an approximation deterministic dynamical systems by a Brownian motion on large space and time scales. In statistical physics such a time-space rescaling often means the transition from the microscopic time scale to the macroscopic one. As a result, a completely deterministic model will asymptotically behave as a Brownian motion.
Thus the distribution of $\eta(t)$ tends to the Gaussian law with the mean (8) and variance (9).
This confirms the diffusion character of $\eta(t)$. It follows that the Eq. (6) (or (1) with a shot noise forcing function) can be used for stochastic and chaotic modeling of the Wiener process.

3.1.2. Power spectral density

Due to (6) the power spectrum of the process $\eta(t)$ is related to the psd of the impulse process $\xi(t)$ by

$$S_\eta(\omega) = \frac{S_\xi(\omega)}{\omega^2}, \qquad (15)$$

where $S_\xi(\omega)$ is defined in [3,4] by

$$\begin{aligned} S_\xi(\omega) &= \frac{\sigma_A^2}{\bar{T}} + \frac{2}{\bar{T}}\sum_{k=1}^{\infty} c_A(k)\cdot \mathrm{Re}\,\Theta_k(\omega) \\ &= \frac{1}{\bar{T}}\cdot \sum_{k=-\infty}^{\infty} c_A(k)\cdot\Theta_k(\omega) \end{aligned} \qquad (16)$$

Here $\Theta_k(\omega) = E\left(\exp\left(j\omega\tau_k\right)\right)$ is the characteristic function of the arrival times τ_k, $\bar{A}$ and $\bar{T}$ are the mean values of the magnitudes and inter-arrival times, respectively.

3.2. Examples

3.2.1. Chaotic shot noise forcing

We first consider a situation where both the magnitudes and the inter-arrival times are generated by chaotic maps. Let A_k be generated by a Bernoulli map defined by

$$\phi(x) = \begin{cases} 2x+1, & -1 \le x < 0 \\ 2x-1, & 0 \le x < 1 \end{cases}.$$

In this case $\{A_k\}$ is stationary with uniform probabilistic measure and a acf of the form (12) with $r = ½$.
Let the inter-arrival times $\{T_k\}$ be a chaotic sequence generated by the tent map over the interval (0,1).
The expression for the characteristic function $\Theta_k(\omega)$ of the arrival time τ_k when the intervals are generated by a tent map is given by

$$\Theta_k(\omega) = \frac{1}{2^{k-1}}\sum_{j=1}^{2^{k-1}} e^{i\omega f(k,j)}\Theta_1\left(\frac{\omega}{2^{k-1}}(2j-1)\right), \qquad (17)$$

where

$$f(k,j) = \begin{cases} f(k-1,j) + \dfrac{2j-1}{2^{k-1}}, & for\;\; j = 1,2,\ldots,2^{k-2} \\ f(k-1, j-2^{k-2}), & for\;\; j = 2^{k-2}+1,\ldots,2^{k-1} \end{cases}$$

with $f(2,1)=1$, $f(2,2)=0$ and $\Theta_1(\omega) = (e^{i\omega}-1)/i\omega$.
We are thus in the position to obtain both the psd of the shot noise process using the formula (16) and the psd of the process $\eta(t)$ from (15). Thesefunctions are depicted in the Fig. 4 and Fig. 5, respectively.

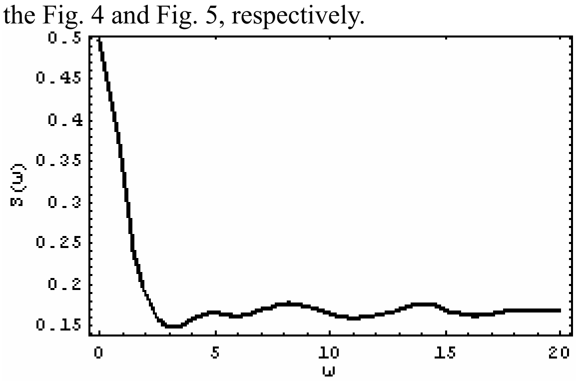

Fig. 4 PSD of the chaotic shot noise

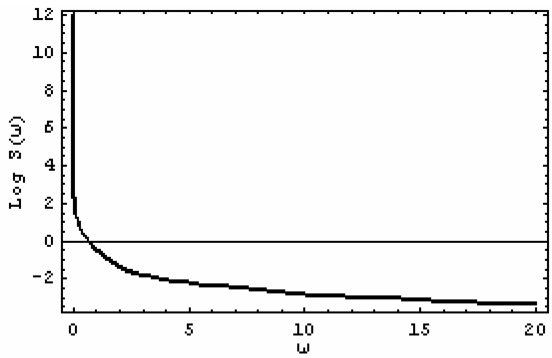

Fig. 5 PSD of the response to the chaotic shot noise forcing

3.2.2. Poisson shot noise forcing

Let the forcing signal be a weakly correlated random/chaotic Poisson shot noise $\xi(t)$. We introduce the following acf of the magnitudes A_k

$$\sigma_A^2 = 2\sigma,\ c_A(1) = -\sigma,\ c_A(k) = 0,\ \forall k \geq 2 \qquad (18)$$

which simplifies (16). We calculate a characteristic function of the arrival times τ_k at lag 1 using the exponentially distributed inter-intervals T_k: $p_T(y) = \alpha e^{-\alpha y}$. Then we get

$$\mathrm{Re}\left(\Theta_1(\omega)\right) = \alpha \int_0^\infty \cos(\omega x) e^{-\alpha x} dx = \frac{\alpha}{\omega^2 + \alpha^2} \qquad (19)$$

Eqs. (18) and (19) substituted into (16) and (15) give

$$S_\eta(\omega) = \frac{2\alpha\sigma}{\omega^2 + \alpha^2} \qquad (20)$$

Notice that the chaotic case the exponentially distributed interarrival times T_k and the magnitudes A_k with the acf (18) can be generated by maps (4) and (5),respectively. Design of such chaotic maps can be done by use the inverse methods [e.g. 5]. The following map is one of the solutions of this inverse problem:

$$\varphi(x) = \begin{cases} -\dfrac{1}{\alpha}\ln\left(2e^{-\alpha x} - 1\right), & 0 < x < \dfrac{\ln 2}{\alpha} \\ -\dfrac{1}{\alpha}\ln\left(1 - 2e^{-\alpha x}\right), & x > \dfrac{\ln 2}{\alpha} \end{cases} \qquad \text{(Fig. 6)}$$

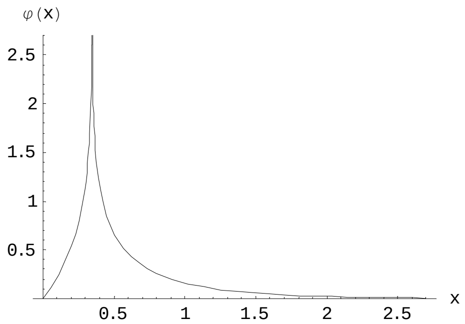

Fig. 6 A map with an exponential distribution

Thus a solution $\eta(t)$ of the differential equation (1) forced by a weakly correlated random/chaotic Poisson shot noise $\xi(t)$ is characterized by Gaussian distribution and Lorentzian type of power spectrum and can be used e.g. for a phase noise modeling in communication systems.

4. Conclusions

In this work we studied a dynamical system described by Eq. (1) subjected to a forcing in the form of a deterministic or a random impulse process, and shown that under generic conditions the response satisfies the central limit theorem.

The main characteristic of Eq. (1) is the absence of drift term in the right hand side. An obvious realization of this situation is that of phase variables. In this respect it is expected that our results will be of interest in electronic circuits and communication systems. It would nevertheless be desirable to extend the analysis in order to account for a non-trivial intrinsic dynamics in the form of e.g. a linear drift or of nonlinear cooperative processes leading to a generalized Langevin-type equation of the form

$$\frac{d\eta}{dt} = f(\eta) + \xi(t) \qquad (21)$$

It is likely that the case of a linear function $f(\eta)$ (cf. Eq. (2)) will be amenable to a comprehensive analytic study, leading to expressions for the power spectrum and the probability density function of the response. On the other hand the conditions of validity of a central limit type theorem are now expected to be much more stringent at least when the forcing is of deterministic origin, since the probability density of such a forcing is typically both non-Gaussian and has a finite support.

Finally, it would be of interest to address the inverse problem of modeling a process described by a time series displaying some well-defined statistical properties, by a Langevin-type of equation of the form of (21). The issue here would be to design $f(\eta)$ and the forcing $\xi(t)$ in such a way that these statistical properties are satisfied. Work in both directions is in progress.

Acknowledgments

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