2005 International Symposium on Nonlinear Theory and its Applications (NOLTA2005)
Bruges, Belgium, October 18-21, 2005

Digital control of a water tank system with hysteresis

Hideyuki Suzuki† and Kazuyuki Aihara‡

†Department of Mathematical Informatics, The University of Tokyo
7–3–1 Hongo, Bunkyo-ku, Tokyo 113–8656, Japan
Email: hideyuki@sat.t.u-tokyo.ac.jp

‡Institute of Industrial Science, The University of Tokyo
4–6–1 Komaba, Meguro-ku, Tokyo 153–8505, Japan
ERATO Aihara Complexity Modelling Project, JST
45–18 Oyama, Shibuya-ku, Tokyo 151–0065, Japan

Abstract—We investigate a very simple water tank system whose water level is digitally controlled with hysteresis. In spite of its simple mechanism, we show that its behavior is very complicated.

1. Introduction

Hysteresis is very useful and common in control, as can be seen in familiar examples such as on-off temperature control, which usually has a deadband (hysteresis) to avoid frequent on-off switching at short intervals [1]. Meanwhile, as computer technology is developed, use of digital control is also becoming very common. Then, what happens if a system with hysteresis is digitally controlled? In this paper, we consider a very simple water tank system whose water level is digitally controlled with hysteresis.

A schematic diagram of the water tank system is shown in Fig. 1. The water level of the tank is controlled with hysteresis to keep it around h_0 and h_1. If state transitions occur exactly at the endpoints of the deadband, this system is very common and the behavior is obvious; the state is attracted into the obvious limit cycle in a finite time and fairly stable.

Our interest in this paper is in the behavior of the system digitally controlled. If the system is digitally controlled, or more specifically, if the decision of the state transition is made at a uniform intervals, state transition does not necessarily occur at the endpoints anymore. In this paper, we will show that this system can be reduced to a double rotation [2], and it shows a complicated behavior.

It should be noted that we consider the system as an abstract model; the water level of the tank can be replaced with anything else (e.g., room temperature), only if the dynamics is preserved. Besides, this system is undoubtedly one of the simplest example of hybrid systems (see Ref. [1] for introduction to hybrid systems).

2. Water-tank system with hysteresis

We consider a water tank that has a controllable incoming flow and a constant outgoing flow as shown in Fig. 1.

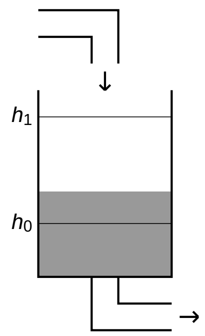

Figure 1: A schematic diagram of the water tank system. The tank has a controllable incoming flow and a constant outgoing flow.

The incoming flow can be turned on and off with a valve: if turned on (v = on), the water level increases at a rate of r_1; if turned off (v = off), the water level decreases at a rate of r_0. Namely, the behavior of the system can be described as follows:

$$\dot{u} = \begin{cases} -r_0 & \text{if } v = \text{off}, \\ +r_1 & \text{if } v = \text{on}, \end{cases} \tag{1}$$

where u denotes the water level of the water tank.

We can observe the water level through two sensors: S_0 and S_1. A sensor can detect if the water level is below the sensor (on) or not (off). Let the sensors S_0 and S_1 are located at the level of h_0 and h_1, respectively, where $h_0 < h_1$. Then, the measurement values from the sensors varies depending on the water level u as follows:

$$(S_0, S_1) = \begin{cases} (\text{off}, \text{off}) & \text{if } u < h_0, \\ (\text{on}, \text{off}) & \text{if } h_0 \le u < h_1, \\ (\text{on}, \text{on}) & \text{if } h_1 \le u. \end{cases}$$

To keep the water level around h_0 and h_1, we control the valve with hysteresis according to the information from the sensors. Specifically, we control the valve with the following rule:

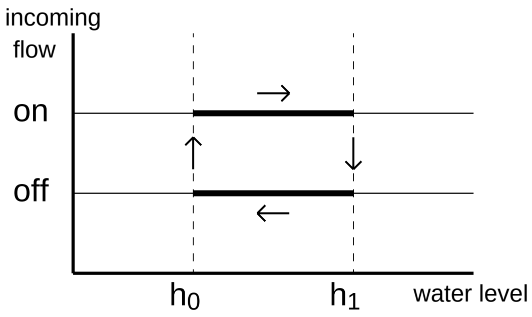

Figure 2: The state space of the water tank system. The thick line denotes the stable limit cycle of the system.

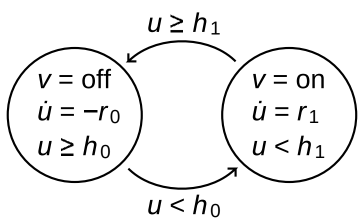

Figure 3: State transitions of the water tank system. The system has a discrete variable v for the state of the valve which can be turned "on" and "off". When v = on, the water level u increases at a rate of r_1, and when v = off, the water level u decreases at a rate of r_0. The state transitions occur at the instants when the water level u escapes out from the deadband $h_0 \leq u < h_1$.

- if S_0 is off, turn on the valve;
- if S_1 is on, turn off the valve;
- otherwise, keep the valve intact (i.e., deadband).

With this control rule, the behavior of the system is deterministic, and therefore it is a (hybrid) dynamical system. As shown in Fig. 2, the state space of the system is $\mathbb{R} \times \{\text{off}, \text{on}\}$, where the first state variable $u \in \mathbb{R}$ denotes the water level, and the second state variable $v \in \{\text{off}, \text{on}\}$ denotes the state of the valve. Then, as shown in Fig. 3, the state evolves according to the following equation in conjunction with Eq. (1):

$$v' = \begin{cases} \text{on} & \text{if } u < h_0 \text{ and } v = \text{off}, \\ \text{off} & \text{if } u \geq h_0 \text{ and } v = \text{off}, \\ \text{on} & \text{if } u < h_1 \text{ and } v = \text{on}, \\ \text{off} & \text{if } u \geq h_1 \text{ and } v = \text{on}. \end{cases} \tag{2}$$

Note that the state transitions occur at the instant when u reaches at the thresholds.

Then, as shown in Fig. 4(a), the behavior of the system is obvious; the state is eventually attracted into the stable limit cycle in a finite time. This kind of on-off control is commonly used in engineering.

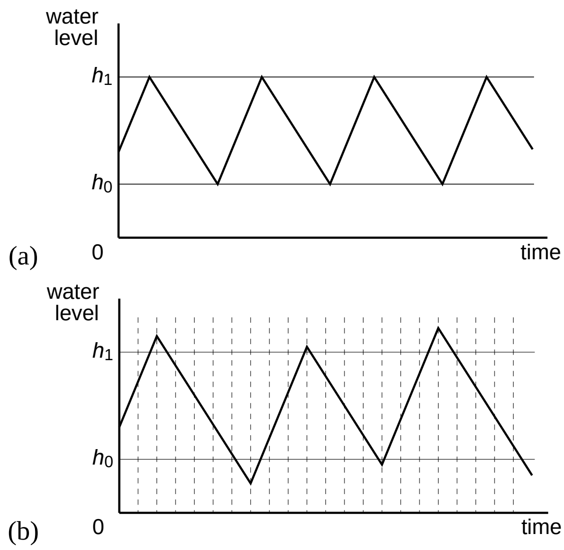

Figure 4: (a) A typical time evolution of the water tank system with hysteresis. Behavior of the system is eventually periodic. (b) Digital control of the water tank system with hysteresis. State transitions occur only at uniform intervals.

3. Digital control

In this section, we are interested in what happens if the water level is digitally controlled. Specifically, we assume that the state transitions described by Eq. (2) occur only at uniform intervals of d seconds.

Let $(u, v) \in \mathbb{R} \times \{\text{off}, \text{on}\}$ be the state of the system at the sampling time t. Then state at the next sampling time $t + d$ can be described by the map $P : \mathbb{R} \times \{\text{off}, \text{on}\} \to \mathbb{R} \times \{\text{off}, \text{on}\}$ as follows:

$$P(u, v) = \begin{cases} (u + r_1 d, \text{on}) & \text{if } u < h_0 \text{ and } v = \text{off}, \\ (u - r_0 d, \text{off}) & \text{if } u \geq h_0 \text{ and } v = \text{off}, \\ (u + r_1 d, \text{on}) & \text{if } u < h_1 \text{ and } v = \text{on}, \\ (u - r_0 d, \text{off}) & \text{if } u \geq h_1 \text{ and } v = \text{on}. \end{cases} \tag{3}$$

A typical time evolution of the digitally controlled water tank system is shown in Fig. 4(b).

Let $I_{\text{off}} = [h_0 - r_0 d, h_0) \times \{\text{off}\}$ and $I_{\text{on}} = [h_1, h_1 + r_1 d) \times \{\text{on}\}$. Then every orbit of the system visits the intervals I_{off} and I_{on} alternately. For a state $(u, \text{off}) \in I_{\text{off}}$, let $n_{\text{on}}(u)$ be the smallest positive integer such that $P^{n_{\text{on}}(u)}(u) \in I_{\text{on}}$. Similarly, for a state $(u, \text{on}) \in I_{\text{on}}$, let $n_{\text{off}}(u)$ be the smallest positive integer such that $P^{n_{\text{off}}(u)}(u) \in I_{\text{off}}$. Specifically, $n_{\text{on}}(u)$ and $n_{\text{off}}(u)$ are given by

$$\begin{aligned} n_{\text{on}}(u) &= -\lfloor (u - h_1)/(r_1 d) \rfloor, \\ n_{\text{off}}(u) &= \lfloor (u - h_0)/(r_0 d) \rfloor + 1. \end{aligned}$$

We also define the maps $P_0 : I_{\text{off}} \to I_{\text{on}}$ and $P_1 : I_{\text{on}} \to I_{\text{off}}$

by

$$P_0: (u, \text{off}) \mapsto (u + n_{\text{on}}(u) r_1 d, \text{on}),$$
$$P_1: (u, \text{on}) \mapsto (u - n_{\text{off}}(u) r_0 d, \text{off}).$$

By reversing the sign of the state variable $u \in \mathbb{R}$ if necessary, we can assume $r_0 \leq r_1$ without any loss of generality. Then the first return map of P on the interval I_{off} is

$$P|_{I_{\text{off}}}(u) = P_1 \circ P_0(u) = u + n_{\text{on}}(u) r_1 d - n_{\text{off}}((P_0(u)) r_0 d.$$

It follows from the assumption $r_0 \leq r_1$ that

$$n_{\text{on}}(u) = \begin{cases} k & \text{if } u < h_1 - (k-1) r_1 d, \\ k-1 & \text{if } u \geq h_1 - (k-1) r_1 d, \end{cases}$$

where $k = \lfloor (h_1 - h_0 + r_0 d + r_1 d)/(r_1 d) \rfloor$. Therefore, we obtain

$$P|_{I_{\text{off}}}(u) = \begin{cases} u + k r_1 d & \text{if } u < h_1 - (k-1) r_1 d, \\ u + (k-1) r_1 d & \text{if } u \geq h_1 - (k-1) r_1 d, \end{cases} \quad (\text{mod } r_0 d)$$

Now, we normalize the interval I_{off} to $[0, 1)$ with the linear map $H(x) = (r_0 d(x-1) + h_0, \text{off})$. Then we have

$$H^{-1} \circ P|_{I_{\text{off}}} \circ H(x) = \begin{cases} \{x + \alpha\} & \text{if } x < c, \\ \{x + \beta\} & \text{if } x \geq c, \end{cases}$$

where $\{\cdot\}$ is defined by $\{x\} = x - \lfloor x \rfloor$. The parameter values $\alpha, \beta,$ and c are as follows:

$$\begin{aligned} \alpha &= \{k r_1 / r_0\}, \\ \beta &= \{(k-1) r_1 / r_0\} \\ c &= ((h_1 - h_0)/d + r_0 + r_1 - k r_1)/r_0. \end{aligned} \tag{4}$$

Note that, if $c \geq 1$, the map is a simple rotation given by $x \mapsto \{x + \alpha\}$.

4. Double rotations

We have shown that the behavior of the water tank system can be reduced to that of the map $f : [0, 1) \to [0, 1)$ defined by

$$f(x) = \begin{cases} \{x + \alpha\} & \text{if } x < c, \\ \{x + \beta\} & \text{if } x \geq c, \end{cases}$$

which is called double rotations [2]. A typical graph of a double rotation is shown in Fig 5(a).

We define the discharge number $q_{(\alpha,\beta,c)}(x)$ of a double rotation $f_{(\alpha,\beta,c)}$ for an initial state $x \in [0, 1)$ as

$$q_{(\alpha,\beta,c)}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{[c,1)}(f^i_{(\alpha,\beta,c)}(x)),$$

if the limit exists, where χ is the characteristic function. Fig. 5(b) shows a graph of $q_{(\alpha,\beta,c)}(x)$ as a function of c,

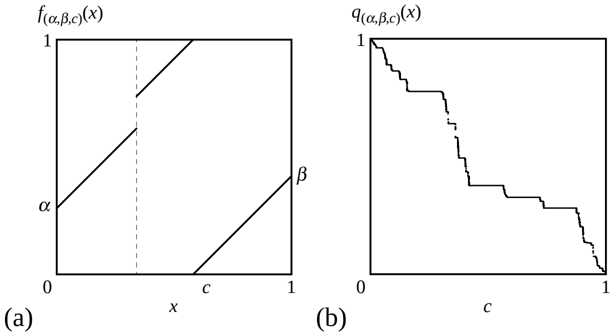

Figure 5: (a) A graph of a double rotation. (b) A discharge number as a function of c for $\alpha \approx 0.281$ and $\beta \approx 0.419$.

where α and β are fixed. The graph is apparently very complicated, resembling a devil's staircase. In fact, for almost every (α, β), there exists a measure-zero Cantor set Γ such that the graph is constant in every connected interval in $[0, 1] \setminus \Gamma$. (See Ref. [2, 4] for details of the self-similar structure in the parameter space of double rotations.)

5. Switching frequency

In this section, we show that the behavior of the water tank system reflects the self-similar structure in the parameter space of double rotations.

If the state is in $[0, c)$ and $[c, 1)$, the input flow is turned on during the duration of exactly kd and $(k-1)d$ seconds, respectively. Therefore, if the water tank system is reduced to a double rotation $f_{(\alpha,\beta,c)}$, the average duration of the input flow is given by $k - q_{(\alpha,\beta,c)}(x)$.

According to Eq. (4), the parameter c depends linearly on $(h_1 - h_0)/d$, and α and β do not depend on $h_1 - h_0$ or d. Therefore, a graph of average duration of the input flow as a function of the deadband width $h_1 - h_0$ (or sampling interval d) resembles a devil's staircase.

The attractor also resembles a Cantor set and has complicated appearance (not shown).

6. Conclusion

We have shown that the behavior of a simple water tank system can be reduced to that of a double rotation. In spite of its simple mechanism, it shows a complicated behavior.

As we pointed out before, this system is an abstract model. This kind of system can be considered as general, as indicated by the fact that the exactly same dynamics arises in a fundamental model of partial discharge phenomena [3].

Since this system is undoubtedly one of the simplest hybrid systems that produce a complicated behavior, it can be expected to be a platform to study a certain class of hybrid systems.

Acknowledgments

The authors thank Dr. Takayuki Nakayama (RIKEN, Japan) for his valuable advice. This work was supported in part by a Grant-in-Aid for Young Scientists (B) No. 15760042 from the Japanese Ministry of Education, Culture, Sports, Science, and Technology.

References

[1] P. J. Antsaklis, "A brief introduction to the theory and applications of hybrid systems," *Proc. IEEE*, vol. 88, pp. 879–887, 2000.

[2] H. Suzuki, S. Ito, and K. Aihara, "Double rotations," *Discrete Contin. Dyn. Syst.*, vol. 13, pp. 515–532, 2005.

[3] H. Suzuki, K. Aihara, and T. Okamoto, "Complex behaviour of a simple partial-discharge model," *Europhys. Lett.*, vol. 66, pp. 28–34, 2004.

[4] H. Bruin and S. Troubetzkoy, "The Gauss map on a class of interval translation mappings," *Israel J. Math.*, vol. 137, pp. 125–148, 2003.