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A Refined Theory for Available Operation of Extremely Complicated Large-Scale Network Systems

Kazuo HORIUCHI†

†Professor Emeritus, Waseda University
3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555 Japan
Email: noriko@oishi.info.waseda.ac.jp

Abstract—In this paper [1], we shall describe about a refined theory based on the concept of set-valued operators, suitable for available operation of extremely complicated large-scale network systems. Fundamental conditions for availability of system behaviors of such network systems are clarified in a form of fixed point theorem for system of set-valued operators.

1. Introduction

In extremely complicated large-scale network systems, precise evaluation and perfect control, and also ideal operation, of overall system behaviors cannot be necessarily expected by using any type of commonplace technologies for maintenance, which might be accomplished by simple measure in usual hierarchical network structures.

In order to effectively evaluate, control and maintain those complicated large-scale networks, as a whole, the author has recommended to introduce some connected-block structure: *i.e.*, whole networks might be separated into several blocks which are carefully self-evaluated, self-controlled and self-maintained by themselves, and so, which are originally self-sustained systems. However, by always carefully watching each other, whenever they observe and detect that some other block is in ill-condition by some accidents, every block can repair and sustain that ill-conditioned block, through inter-block connections, at once. This style of maintenance of system is sometimes called as locally autonomous, but the author recommends that only the ultimate responsibility on observation and regulation of whole system might be left for headquarter itself, which is organized over all blocks just as United States Government [1].

Here, let us consider complete metric linear spaces X_i $(i = 1, \cdots, n)$ and Y_j $(j = 1, \cdots, n)$, and their bounded convex closed subsets $X_i^{(0)}$ and $Y_j^{(0)}$, respectively, corresponding to each block, B_i and B_j of whole network system. Let us introduce operators $f_{ij} : X_i \to Y_j$ such that $f_{ij}(X_i^{(0)}) \subset Y_j^{(0)}$ and let f_{ij} be completely continuous on $X_i^{(0)}$.

For general situations of mutual connections between blocks, by newly introducing n composition-type operators $g_i : X_i \times \Pi_j^n Y_j \times \Pi^n Y_i \to X_i$, where $\Pi_j^n Y_j$ means the direct product of n Y_j's for all $j \in \{1, \cdots, n\}$, and $\Pi^n Y_i$ means the direct product of n Y_i's for fixed i, we have a general system of operator equations:

$$x_i = g_i(x_i;\ f_{i1}(x_i), \cdots, f_{in}(x_i);\ f_{1i}(x_1), \cdots, f_{ni}(x_n)),\quad (i = 1, \cdots, n). \tag{1}$$

Here, we can present a fixed point theorem for this general system of nonlinear operator equations [2], which is an extension of the work by Melvin [3].

However, the fluctuation imposed on the actual system is nondeterministic rather than deterministic. In this case, even the effect due to a single cause is multi-valued, and the behavior is more naturally represented by a set of points, rather than a single point.

Therefore, it is reasonable to consider some suitable-subset of the range of system behavior, in place of single ideal point, as target which the behavior must reach under influence of system control. Now, we can name it as an "available range" or a "tolerable range" of the system behavior. Thus, by the available or tolerable range, we mean the range of behavior, in which every behavior effectively satisfies good conditions beforehand specified, as a set of ideal behaviors. From such a point of view, the theory for fluctuation imposed on the system should be developed concerning the set-valued operator.

By the set-valued operator G defined on a space X is meant a correspondence in which a set $G(x)$ is specified in correspondence to any point x in X. In particular, when $G(X) \subset X$, and if there exists a point x^* such that $x^* \in G(x^*)$, x^* is called a fixed point of G.

The author has given a series of studies on set-valued operators in functional analysis aspects, and has vigorously applied it to analysis of uncertain fluctuations of network systems [4], [5].

Recently, the author gave a general type of fixed point theorem for the system of set-valued operator equations, in order to treat with extremely complicated large-scale network systems [6].

Namely, let us introduce n set-valued operators $G_i : X_i \times \Pi_j^n Y_j \times \Pi^n Y_i \to \mathcal{F}(X_i)$ (the family of all non-empty closed compact subsets of X_i) $(i = 1, \cdots, n)$, where $\Pi_j^n Y_j$ means the direct product of n Y_j's, for any $j \in \{1, \cdots, n\}$, and $\Pi^n Y_i$ means direct product of n Y_i's, for fixed i.

[1] The LaTeX-description of manuscript is made by Dr. K. Maruyama, Visiting Lecturer, Waseda University.

Under some natural conditions, the author presented an important fixed point theorem on the system of set-valued operator equations:

$$x_i \in G_i(x_i;\, f_{i1}(x_i), \cdots,\, f_{in}(x_i);\, f_{1i}(x_1), \cdots,\, f_{ni}(x_n)), \quad (i = 1, \cdots, n). \tag{2}$$

2. A Refined Fixed Point Theorem For System of Set-Valued Operators

Here, we will present a refined theory of the fixed point theorem for such a general system of set-valued operator equations.

For the first step, let us introduce reflexive, or uniformly convex, real Banach spaces X_i $(i = 1, \cdots, n)$, in which the norm is represented by $\|\cdot\|$, and also their non-empty bounded closed convex subsets $X_i^{(0)}$ $(i = 1, \cdots, n)$. Let X_i' be the dual space of X_i and let us introduce a weak topology $\sigma(X_i, X_i')$ into X_i. Then, X_i is locally convex topological linear space, and therefore, $X_i^{(0)}$ is weakly closed and weakly compact. Further, let us consider another real Banach spaces Y_j $(j = 1, \cdots, n)$ in which the norm is represented by $\|\cdot\|$ [7].

Now, let us introduce a series of assumptions:

Assumption 1 *Let the operator $f_{ij} : X_i^{(0)} \to f_{ij}(X_i^{(0)}) \subset Y_j$ be completely continuous in the sense that when a weakly convergent net $\{x_i^\nu\}$ ($\nu \in J$: a directive set) weakly converges to $\overline{x_i}$, then the sequence $\{f_{ij}(x_i^\nu)\}$ has a subsequence which strongly converges to $f_{ij}(\overline{x_i})$ in Y_j.*

Assumption 2 *Let the set-valued operator $G_i : X_i^{(0)} \times \prod_j^n Y_j \times \prod^n Y_j \to \mathcal{F}(X_i)$ (a family of all non-empty closed compact subset of X_i) satisfies the following Lipschitz condition with respect to the Hausdorff distance d_H: that is, there are three kinds of constants $0 < a_i < 1$ and $b_{ji} > 0$, $c_{ji} > 0$ such that for any $x_i^{(1)}, x_i^{(2)} \in X_i$, for any $y_{ji}^{(1)}, y_{ji}^{(2)} \in Y_i$, and for any $y_{ij}^{(1)}, y_{ij}^{(2)} \in Y_j$, G_i's satisfy inequalities:*

$$\begin{aligned} &d_H(G_i(x_i^{(1)}; y_{i1}^{(1)}, \cdots, y_{in}^{(1)}; y_{1i}^{(1)}, \cdots, y_{ni}^{(1)}),\\ &\quad G_i(x_i^{(2)}; y_{i1}^{(2)}, \cdots, y_{in}^{(2)}; y_{1i}^{(2)}, \cdots, y_{ni}^{(2)}))\\ &\le a_i \cdot \|x_i^{(1)} - x_i^{(2)}\| + \textstyle\sum_{j=1}^n b_{ji} \cdot \|y_{ij}^{(1)} - y_{ij}^{(2)}\|\\ &\quad + \textstyle\sum_{j=1}^n c_{ji} \cdot \|y_{ji}^{(1)} - y_{ji}^{(2)}\|, \quad (i = 1, \cdots, n). \end{aligned} \tag{3}$$

Here, the Hausdorff distance d_H between two sets S_1 and S_2 is defined by

$$d_H(S_1, S_2) \triangleq \max\{\sup\{d(x_1, S_2) | x_1 \in S_1\}, \sup\{d(x_2, S_1) | x_2 \in S_2\}\},$$

where $d(x, S) \triangleq \inf\{\|x - y\| \mid y \in S\}$ is the distance between a point x and a set S.

Assumption 3 *For any $x_i \in X_i^{(0)}$ and $y_{ij} \triangleq f_{ij}(x_i)$, $y_{ji} \triangleq f_{ji}(x_j)$, $G_i^{(0)}(x_i;\ y_{i1}, \cdots, y_{in};\ y_{1i}, \cdots, y_{ni}) \triangleq G_i(x_i;\ y_{i1}, \cdots, y_{in};\ y_{1i}, \cdots, y_{ni}) \cap X_i^{(0)} \neq \phi$. Moreover, there exist projection points $\tilde{x}_i'$ of arbitrary point x_i' upon the set $G_i(x_i; y_{i1}, \cdots, y_{in}; y_{1i}, \cdots, y_{ni})$ such that*

$$\begin{aligned} &\|\tilde{x}_i' - x_i'\|\\ &\quad = \min\{\|x_i' - z_i\| \mid z_i \in G_i(x_i; y_{i1}, \cdots, y_{in}; y_{1i}, \cdots, y_{ni})\}. \end{aligned} \tag{4}$$

Then, we have an important lemma:

Lemma 1 *For all i $(i = 1, \cdots, n)$, let us adopt arbitrary points $x_i \equiv z_i^{(0)} \in X_i^{(0)}$ and also fix all values of $f_{ij}(z_i^{(0)})$ and $f_{ji}(z_j^{(0)})$, $(j = 1, \cdots, n)$. Now, for every i, let us introduce a sequence $\{z_i^k\}$ $(k = 0, 1, 2, \cdots)$, starting from the above-adopted point $z_i^{(0)}$, and with each $z_i^k \in X_i^{(0)}$ as a projection point of $z_i^{k-1} \in X_i^{(0)}$ upon the set $G_i(z_i^{k-1};\ f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)});\ f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)}))$. Then, this sequence $\{z_i^k\}$ $(k = 0, 1, 2, \cdots)$ is a Cauchy sequence, having its own limit points $\overline{z_i} \in X_i^{(0)}$, such that*

$$\overline{z_i} \in G_i^{(0)}(\overline{z_i};\, f_{i1}(z_i^{(0)}), \cdots,\, f_{in}(z_i^{(0)});\, f_{1i}(z_1^{(0)}), \cdots,\, f_{ni}(z_n^{(0)}), \quad (i = 1, \cdots, n)\,. \tag{5}$$

(Proof is in Appendix A.)

Here, the correspondence from the starting point $z_i^{(0)} \equiv x_i \in X_i^{(0)}$ to the limit points $\overline{z_i} \in X_i^{(0)}$ is multivalued, in general, and hence, by this correspondence we can define a set-valued operator $H_i : x_i \to \{\overline{z_i}\}$ in X_i; *i.e.*, $\overline{z_i} \in H_i(x_i)$. If this set-valued operator H_i has a fixed point x_i^*: *i.e.*, $x_i^* \in H_i(x_i^*)$, then it satisfies the system of equations:

$$x_i^* \in G_i^{(0)}(x_i^*;\, f_{i1}(x_i^*), \cdots,\, f_{in}(x_i^*);\, f_{1i}(x_1^*), \cdots,\, f_{ni}(x_n^*)), \quad (i = 1, \cdots, n) \tag{6}$$

by Eq. (5). Therefore, x_i^* is the solution of the system of set-valued operator equations (2).

Now, let us refer to the well-known fixed point theorem for set-valued operator:

Lemma 2 (Ky Fan [8]) *Let X_i be a locally convex topologica linear space, and $X_i^{(0)}$ be a non-empty convex compact subset of X_i. Let $\mathcal{H}_c(X_i^{(0)})$ be the family of all non-empty closed convex subset of $X_i^{(0)}$. Then, for any upper semicontinuous set-valued operator H_i: $X_i^{(0)} \to \mathcal{H}_c(X_i^{(0)})$, there exists a fixed point $x_i^* \in X_i^{(0)}$ such that $x_i^* \in H_i(x_i^*)$.*

In order to apply this lemma to our problem, we must verify that the above-defined set-valued operator $H_i(x_i)$ is upper semicontinuous, and its range is closed and convex.

In the first place, the closedness of the range of $H_i(x_i)$ is easily verified from the assumption 2. (Proof is in Appendix B.)

For the verification of the convexity, it is sufficient to add the following assumption:

Assumption 4 (Rockafellar [9]) *For any $x_i^{(1)}, x_i^{(2)}, \in X_i^{(0)}$, and for any constant $r(0 < r < 1)$, uniformly with respect to every $y_{ij} \in Y_j$ and $y_{ji} \in Y_i$, G_i satisfies the relation:*

$$\begin{aligned} & r \cdot G_i(x_i^{(1)};\ y_{i1}, \cdots, y_{in};\ y_{1i}, \cdots, y_{ni}) \\ & \quad +(1-r) \cdot G_i(x_i^{(2)};\ y_{i1}, \cdots, y_{in};\ y_{1i}, \cdots, y_{ni}) \\ & \quad \subset G_i(r \cdot x_i^{(1)} + (1-r) \cdot x_i^{(2)};\ y_{i1}, \cdots, y_{in};\ y_{1i}, \cdots, y_{ni})\ . \end{aligned} \tag{7}$$

In fact, under the assumption 4, we have

$$\begin{aligned} & r \cdot z_i^{(1)} + (1-r) \cdot z_i^{(2)} \in r \cdot G_i(z_i^{(1)};\ y_{i1}, \cdots, y_{in};\ y_{1i}, \cdots, y_{ni}) \\ & \quad +(1-r) \cdot G_i(z_i^{(2)};\ y_{i1}, \cdots, y_{in};\ y_{1i}, \cdots, y_{ni}) \\ & \quad \subset G_i(r \cdot z_i^{(1)} + (1-r) \cdot z_i^{(2)};\ y_{i1}, \cdots, y_{in};\ y_{1i}, \cdots, y_{ni}), \end{aligned} \tag{8}$$

for any $z_i^{(\nu)} \in H_i(x_i)$ $(\nu = 1, 2)$: *i.e.*, for any $z_i^{(\nu)} \in G_i(z_i^{(\nu)};\ f_{i1}(x_i), \cdots, f_{in}(x_i);\ f_{1i}(x_1), \cdots, f_{ni}(x_n))$ $(\nu = 1, 2)$. This relation means the convexity of $H_i(x_i)$.

Lastly, in order to verify the upper semicontinuity, we should prove that if an arbitrary weakly convergent net $\{x_i^\nu\}(\nu \in J)$ in $X_i^{(0)}$ weakly converges to $\overline{x}_i$ and if the weakly convergent net $\{z_i^\nu\}$ $(\nu \in J)$ in $X_i^{(0)}$ made from $z_i^\nu \in H_i(x_i^\nu)$ weakly converges to $\overline{z}_i$, we have $\overline{z}_i \in H_i(\overline{x}_i)$. For this purpose, we can use the following lemma:

Lemma 3 (Nadler [10]) *Let X_i be complete metric space, and let F_i^ν $(\nu \in J)$ and $\overline{F}_i : X_i \to \mathcal{F}_c(X_i)$ (the family of all non-empty closed compact subsets of X_i) be set-valued operators contracting with respect to the Hausdorff distance d_H: e.g., there exists a constant a_i $(0 < a_i < 1)$ such that for any $z_i^{(1)}, z_i^{(2)} \in X_i$, F_i^ν satisfies the inequality*

$$d_H(F_i^\nu(z_i^{(1)}), F_i^\nu(z_i^{(2)})) \le a_i \cdot \|z_i^{(1)} - z_i^{(2)}\|\ . \tag{9}$$

Now, let $\{F_i^\nu\}$ be uniformly convergent to $\overline{F}_i$ in the distance d_H. Let z_i^ν be a fixed point of F_i^ν. Then, we can find that the sequence $\{z_i^\nu\}$ $(\nu \in J)$ has a convergent subsequence $\{z_i^{\nu m}\}$ and its limit point $\vec{z}_i$ is a fixed point of $\overline{F}_i$: $\vec{z}_i \in \overline{F}_i(\vec{z}_i)$.

From the assumption 1, we remember that when any weakly convergent net $\{x_i^\nu\}$ $(\nu \in J)$ weakly converges to $\overline{x}_i$, the net $\{f_{ij}(x_i^\nu)\}$ has a subsequence $\{f_{ij}(x_i^{\nu m})\}$ strongly convergent to $f_{ij}(\overline{x}_i)$. On the other hand, from the assumption 2, we have

$$\begin{aligned} & \sup_{x_i \in X_i} d_H(G_i(x_i;\ f_{i1}(x_i^{\nu m}), \cdots, f_{in}(x_i^{\nu m}); \\ & \qquad f_{1i}(x_1^{\nu m}), \cdots, f_{ni}(x_n^{\nu m})), \\ & \quad G_i(x_i;\ f_{i1}(\overline{x}_i), \cdots, f_{in}(\overline{x}_i);\ f_{1i}(\overline{x}_1), \cdots, f_{ni}(\overline{x}_n)) \\ & \le \sum_{j=1}^n b_{ji} \| f_{ij}(x_i^{\nu m}) - f_{ij}(\overline{x}_i) \| \\ & \quad + \sum_{j=1}^n c_{ji} \| f_{ji}(x_j^{\nu m}) - f_{ji}(\overline{x}_j) \| \to 0. \end{aligned} \tag{10}$$

This implies that the sequence of the set-valued operators

$$\{G_i(x_i;\ f_{i1}(x_i^{\nu m}), \cdots, f_{in}(x_i^{\nu m});\ f_{1i}(x_1^{\nu m}), \cdots, f_{ni}(x_n^{\nu m}))\}$$

uniformly converges to

$$G_i(x_i;\ f_{i1}(\overline{x}_i), \cdots, f_{in}(\overline{x}_i);\ f_{1i}(\overline{x}_1), \cdots, f_{ni}(\overline{x}_n))$$

in the distance d_H. Thus, from this deduction, substituting $F_i^\nu, F_i^{\nu m}$ and $\overline{F}_i$ by

$$\begin{aligned} & G_i(z_i;\ f_{i1}(x_i^\nu), \cdots, f_{in}(x_i^\nu);\ f_{1i}(x_1^\nu), \cdots, f_{ni}(x_n^\nu)), \\ & \quad G_i(z_i;\ f_{i1}(x_i^{\nu m}), \cdots, f_{in}(x_i^{\nu m});\ f_{1i}(x_1^{\nu m}), \cdots, f_{ni}(x_n^{\nu m})) \end{aligned}$$

and

$$G_i(z_i;\ f_{i1}(\overline{x}_i), \cdots, f_{in}(\overline{x}_i);\ f_{1i}(\overline{x}_1), \cdots, f_{ni}(\overline{x}_n)),$$

respectively, we can apply the Lemma 3, and hence, we find that the sequence of fixed points

$$\begin{aligned} \{\overline{z}_i^{\nu m}\} :\ & \overline{z}_i^{\nu m} \in G_i(\overline{z}_i^{\nu m};\ f_{i1}(x_i^{\nu m}), \cdots, f_{in}(x_i^{\nu m}); \\ & f_{1i}(x_1^{\nu m}), \cdots, f_{ni}(x_n^{\nu m})), \end{aligned}$$

i.e., $\overline{z}_i^{\nu m} \in H_i(x_i^{\nu m})$, strongly, and therefore, weakly converges to the fixed point

$$\vec{z}_i : \vec{z}_i \in G_i(\vec{z}_i;\ f_{i1}(\overline{x}_i), \cdots, f_{in}(\overline{x}_i);\ f_{1i}(\overline{x}_1), \cdots, f_{ni}(\overline{x}_n)),$$

i.e., $\vec{z}_i \in H_i(\overline{x}_i)$.
As a result, we have a theorem:

Theorem 1 *Let X_i be a reflexive, or uniformly convex, real Banach space, and $X_i^{(0)}$ be a non-empty bounded closed convex subset of X_i. By the dual space $X_i^{'}$, let us introduce a weak topology $\sigma(X_i, X_i^{'})$ into X_i. Let f_{ij} and G_i be deterministic and set-valued operators, respectively, which satisfy the series of assumptions 1 to 4. Then, we have a Cauchy sequence $\{z_i^k\} \subset X_i^{(0)}$ $(k = 0, 1, 2, \cdots)$, introduced by the succesive procedure in Lemma 1. This sequence has a set of limit points $\{\overline{z}_i\}$, and we can define a set-valued operator H_i by the correspondence from the arbitrary starting point $z_i^{(0)} \equiv x_i \in X_i^{(0)}$ to the set of limit points $\{\overline{z}_i\}$ in $X_i^{(0)}$: $\overline{z}_i \in H_i(x_i)$. This set-valued operator H_i has a fixed point x^* in $X_i^{(0)}$, which is, in turn, the solution of the system of set-valued operator equations (2).*

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Appendix

A. The Cauchy Sequence $\{z_i^k\}$ and Its Limit Point $\overline{z}_i$

For any integers $k \geq 0$ and $j \geq 0$ $(k + j \geq 1)$, we have

$$\begin{aligned}
&\|z_i^{k+j} - z_i^{k+j+1}\| \\
&\quad \equiv d(z_i^{k+j}, G_i^{(0)}(z_i^{k+j}; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); \\
&\qquad f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)}))) \\
&\quad \leq d_H(G_i^{(0)}(z_i^{k+j-1}; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); \\
&\qquad f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)}), \\
&\qquad G_i^{(0)}(z_i^{k+j}; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); \\
&\qquad f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)}))) \\
&\quad \leq a_i \cdot \|z_i^{k+j-1} - z_i^{k+j}\| \\
&\quad \cdots \\
&\quad \leq a_i^{k+j} \cdot \|z_i^{(0)} - z_i^1\| \ .
\end{aligned}$$

Therefore, for any $k \geq 0$ and $m \geq 0$, we find

$$\begin{aligned}
\|z_i^k - z_i^{k+m}\| &\leq \textstyle\sum_{j=0}^{m-1} \|z_i^{k+j} - z_i^{k+j+1}\| \\
&\leq \|z_i^{(0)} - z_i^1\| \cdot \textstyle\sum_{j=0}^{m-1} a_i^{k+j} \\
&\leq \|z_i^{(0)} - z_i^1\| \cdot \textstyle\sum_{j=0}^{\infty} a_i^{k+j} \\
&= a_i^k \cdot (1 - a_i)^{-1} \cdot \|z_i^{(0)} - z_i^1\| \ .
\end{aligned}$$

Hence, $\|z_i^k - z_i^{k+m}\| \to 0$, as $k \to +\infty$, *i.e.*, the sequence $\{z_i^k\}$ is a Cauchy sequence. Since X_i is complete and $X_i^{(0)}$ is closed, $\{z_i^k\}$ has at least one limit point $\overline{z}_i \in X_i^{(0)}$.

Now, let us consider a projection point $\hat{z}_i$ of z_i^{k+1} upon $G_i^{(0)}(\overline{z}_i;\ f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)});\ f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)}))$. Then, we have

$$\begin{aligned}
&d(\overline{z}_i, G_i^{(0)}(\overline{z}_i; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)}))) \\
&\quad \leq \|\overline{z}_i - \hat{z}_i\| \\
&\quad \leq \|\overline{z}_i - z_i^k\| + \|z_i^k - z_i^{k+1}\| + \|z_i^{k+1} - \hat{z}_i\| \ .
\end{aligned}$$

On the other hand, we see

$$\begin{aligned}
&\|z_i^{k+1} - \hat{z}_i\| \\
&\quad \equiv d(z_i^{k+1}, G_i^{(0)}(\overline{z}_i; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); \\
&\qquad f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)}))) \\
&\quad \leq d_H(G_i^{(0)}(z_i^k; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)})), \\
&\qquad G_i^{(0)}(\overline{z}_i; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)}))) \\
&\quad \leq a_i \cdot \|z_i^k - \overline{z}_i\| \ .
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&d(\overline{z}_i, G_i^{(0)}(\overline{z}_i; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)}))) \\
&\quad \leq (1 + a_i) \cdot \|z_i^k - \overline{z}_i\| + \|z_i^k - z_i^{k+1}\| \to 0 \ ,
\end{aligned}$$

as $k \to \infty$. Since the left-hand side is independent of k, $d(\overline{z}_i, G_i^{(0)}(\overline{z}_i; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)}))) \equiv 0$, and so, we have the relation (5).

B. Closedness of the Range of $H_i(x_i)$

Let us consider a sequence $\{z_i^k\}$ $(k = 0, 1, 2, \cdots)$ such that $z_i^k \in H_i(x_i)$, *i.e.*,

$$z_i^k \in G_i^{(0)}(z_i^k; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_1^{(0)}))$$

and let it strongly converge to a limit point $\overline{z}_i$.

Then, we have

$$\begin{aligned}
&d(z_i^k, G_i^{(0)}(\overline{z}_i; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)}))) \\
&\quad \triangleq \inf\{\|z_i^k - z_i\| \mid z_i \in G_i^{(0)}(\overline{z}_i; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); \\
&\qquad f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)}))\} \\
&\quad \leq d_H(G_i^{(0)}(z_i^k; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); \\
&f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)})), G_i^{(0)}(\overline{z}_i; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); \\
&\qquad f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)}))) \\
&\quad \leq a_i \cdot \|z_i^k - \overline{z}_i\| \ .
\end{aligned}$$

Now, let $\hat{z}_i$ be a projection point, in $X_i^{(0)}$, of z_i^k upon the set $G_i^{(0)}(\overline{z}_i; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)}))$. Then, we find

$$\begin{aligned}
&d(\overline{z}_i, G_i^{(0)}(\overline{z}_i; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)}))) \\
&\quad \leq \|\overline{z}_i - \hat{z}_i\| \\
&\quad \leq \|\overline{z}_i - z_i^k\| + \|z_i^k - \hat{z}_i\| \\
&\quad \leq \|\overline{z}_i - z_i^k\| + d(z_i^k, G_i^{(0)}(\overline{z}_i; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); \\
&\qquad f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)}))) \\
&\quad \leq \|\overline{z}_i - z_i^k\| + a_i \cdot \|z_i^k - \overline{z}_i\| \\
&\quad = (1 + a_i) \cdot \|z_i^k - \overline{z}_i\| \to 0 \ .
\end{aligned}$$

as $k \to 0$. Hence,

$$d(\overline{z}_i, G_i^{(0)}(\overline{z}_i;\ f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)});\ f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)}))) \equiv 0,$$

and so,

$$\overline{z}_i \in G_i(\overline{z}_i; f_{i1}(z_i^{(0)}), \cdots, f_{in}(z_i^{(0)}); f_{1i}(z_1^{(0)}), \cdots, f_{ni}(z_n^{(0)})),$$

which is equivalent to $\overline{z}_i \in H_i(x_i)$.