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Decomposition of Bifurcation Diagram for Periodic Oscillation using Ideal Quotient

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Abstract—This paper presents a method to decompose the bifurcation diagram for periodic oscillation in global parameter space. As a tool of the decomposition, we use ideal decomposition corresponding to algebraic factorization. In order to realize the ideal decomposition, we propose the ideal quotient based on the symmetry of systems and other efficient methods using the ideal quotient partially. Further, we clarify the symmetries of the homogeneous equation in the harmonic balance method and provide a method for systematic decomposition.

1. Introduction

The advance in computer algebra system has led to remarkable progress in applications of Gröbner base [1, 2]. We can find several applications to the analysis of nonlinear circuit systems [3, 4, 5]. As far as the local bifurcation is concerned, the analysis using Gröbner base is reported [3, 4]. However, as for the global bifurcation very few results have been shown in [5, 6].

If our target systems are represented by algebraic equations, we can obtain the bifurcation diagram in global parameter space by using the Gröbner base of lexicographic order [6, 7]. Using the algebraic representation of bifurcation diagram, the reports [6, 7] decompose the global bifurcation diagram into sub-diagrams using algebraic factorization The method provides the mode decompositions of nonlinear systems using ideal decomposition. The approach makes clear also the relation between the symmetry and bifurcations for nonlinear systems.

The purpose of this paper is to apply the method to periodic oscillations on nonlinear circuit systems, and to decompose the bifurcation diagram in global parameter space. We use the harmonic balance method to obtain algebraic determining equation of the periodic oscillations. Using the approach we can decompose the bifurcation diagram by the ideal decomposition. Further the mode decomposition reveals the symmetries which the periodic oscillations have.

In order to obtain good approximations of the periodic oscillations, we have to consider many frequency components in the harmonic balance method. However, the difficulty lies in the increase of the complexity of the computation of Gröbner base of lexicographic order when the number of variables and equations increases. To surpass its limitation, we propose an efficient method of the ideal decomposition by partially applying the ideal quotient based on known symmetries. Although we first use known symmetries for the decomposition, we finally obtain the complete decomposition of the bifurcation diagram by factorizations. Further, we propose a systematic ideal decomposition method for the periodic oscillations using a homogeneous equation in the harmonic balance method.

2. Harmonic Balance Method

We consider a nonlinear circuit equation of n_u dimensions

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\tau} = \boldsymbol{h}(\boldsymbol{u} \mid \boldsymbol{\lambda}) + \boldsymbol{e}(\tau), \tag{1}$$

$$\boldsymbol{h} = (h_1, \dots, h_{n_u})' \in \boldsymbol{R}^{n_u},\ \boldsymbol{e} = (e_1(\tau), \dots, e_{n_u}(\tau))' \in \boldsymbol{R}^{n_u},$$
$$\boldsymbol{u} = (u_1, \dots, u_{n_u})' \in \boldsymbol{R}^{n_u},\ \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_l) \in \boldsymbol{R}^l,$$

where $\boldsymbol{u}$ is a state variable vector, $\boldsymbol{e}(\tau)$ is a source vector of period $2\pi/m$ ($m \in \boldsymbol{Z}_{>0}$), $\boldsymbol{\lambda}$ is a set of circuit parameters, $(\cdot)'$ denotes the transposition, $\boldsymbol{h} : \boldsymbol{R}^{n_u} \times \boldsymbol{R}^l \to \boldsymbol{R}^{n_u}$ is a vector of nonlinear functions which consists of polynomials of $u_1, \dots, u_{n_u}$ with coefficients in rational functions of $\lambda_1, \dots, \lambda_l$ and $\boldsymbol{R}, \boldsymbol{Z}_{>0}$ are the set of real numbers and positive integers, respectively.

We assume that Eq.(1) has a periodic solution $\boldsymbol{u}(\tau)$ defined by

$$\boldsymbol{u}(\tau) = \sum_{j=0}^{\infty} \left\{ \boldsymbol{\psi}_{cj} \cos j\tau + \boldsymbol{\psi}_{sj} \sin j\tau \right\}, \tag{2}$$

where $\boldsymbol{\psi}_{cj} = (\psi_{cj,1}, \dots, \psi_{cj,n_u})' \in \boldsymbol{R}^{n_u}$, $\boldsymbol{\psi}_{sj} = (\psi_{sj,1}, \dots, \psi_{sj,n_u})' \in \boldsymbol{R}^{n_u}$, $\boldsymbol{\psi}_{s0} = \boldsymbol{0}$. We assume that the above solution can be approximated by a truncated Fourier series with frequency component set $\kappa \subset \boldsymbol{Z}_{\geq 0}$,

$$\boldsymbol{u}^*(\tau) = K^* \boldsymbol{u}(\tau) = \sum_{j \in \kappa} \left\{ \boldsymbol{\psi}_{cj} \cos j\tau + \boldsymbol{\psi}_{sj} \sin j\tau \right\}, \tag{3}$$

where K^* is a projection operator that expresses the truncation of the Fourier series; $\boldsymbol{u}^* = (u_1^*, \dots, u_{n_u}^*)' \in \boldsymbol{R}^{n_u}$, and $\boldsymbol{Z}_{\geq 0}$ denotes the set $0, 1, 2, \dots$. The substitution of Eq.(3) into Eq.(1) gives

$$\sum_{j \in \kappa} \left[j\psi_{sj,i} - p_{j,i}(\boldsymbol{\psi} \mid \boldsymbol{\lambda}) + e_{pj,i} \right] \cos j\tau + \sum_{j \in \kappa} \left[-j\psi_{cj,i} - q_{j,i}(\boldsymbol{\psi} \mid \boldsymbol{\lambda}) + e_{qj,i} \right] \sin j\tau = 0, \tag{4}$$

where $p_{0,i}(\boldsymbol{\psi} \mid \boldsymbol{\lambda}) = \frac{1}{2\pi} \int_0^{2\pi} h_i(\boldsymbol{u}^*(\tau) \mid \boldsymbol{\lambda}) \mathrm{d}\tau$,

$$p_{j,i}(\boldsymbol{\psi} \mid \boldsymbol{\lambda}) = \frac{1}{\pi}\int_0^{2\pi} h_i(\boldsymbol{u}^*(\tau) \mid \boldsymbol{\lambda}) \cos j\tau \mathrm{d}\tau \quad (j \neq 0),$$
$$q_{j,i}(\boldsymbol{\psi} \mid \boldsymbol{\lambda}) = \frac{1}{\pi}\int_0^{2\pi} h_i(\boldsymbol{u}^*(\tau) \mid \boldsymbol{\lambda}) \sin j\tau \mathrm{d}\tau,$$
$$\boldsymbol{\psi} = \left\{(\psi_{cj,i}, \psi_{sj,i}) \mid j \in \kappa,\ i = 1, \ldots, n_u\right\},$$
$$e_{p0,i}(\tau) = \frac{1}{2\pi}\int_0^{2\pi} e_i(\tau)\mathrm{d}\tau,$$
$$e_{pj,i}(\tau) = \frac{1}{\pi}\int_0^{2\pi} e_i(\tau) \cos j\tau \mathrm{d}\tau \quad (j \neq 0),$$
$$e_{qj,i}(\tau) = \frac{1}{\pi}\int_0^{2\pi} e_i(\tau) \sin j\tau \mathrm{d}\tau,$$
$$i = 1, \ldots, n_u.$$

From Eq.(4), We obtain algebraic equations of harmonic balance

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{\psi}) &= \boldsymbol{f}(\boldsymbol{\psi} \mid \boldsymbol{\lambda}, \hat{\boldsymbol{e}}) \\ &\equiv \begin{bmatrix} j\psi_{sj,i} - p_{j,i}(\boldsymbol{\psi} \mid \boldsymbol{\lambda}) + e_{pj,i} \\ -j\psi_{cj,i} - q_{j,i}(\boldsymbol{\psi} \mid \boldsymbol{\lambda}) + e_{qj,i} \end{bmatrix} = \boldsymbol{0}, \end{aligned} \tag{5}$$
$$i = 1, \ldots, n_u,\quad j \in \kappa,\quad \hat{\boldsymbol{e}} \equiv \begin{bmatrix} e_{pj,i} \\ e_{qj,i} \end{bmatrix} \in \boldsymbol{R}^n,\quad n = |\boldsymbol{\psi}|,$$

where $|\cdot|$ denotes the number of the element of set.

3. Ideal Decomposition with Ideal Quotient

3.1. Decomposition of Ideal

The real variety $\boldsymbol{V}[\boldsymbol{f}(\boldsymbol{\psi})]$ is defined by the set of all real solutions of Eq.(5). The fact shows that the bifurcation diagram of Eq.(5), depicting $\boldsymbol{\psi}$ versus $(\boldsymbol{\lambda}, \hat{\boldsymbol{e}})$, corresponds to the real variety $\boldsymbol{V}[\boldsymbol{f}(\boldsymbol{\psi})]$. In order to decompose the variety $\boldsymbol{V}[\boldsymbol{f}(\boldsymbol{\psi})]$, we can use ideal decomposition [1, 2]. That is, if the ideal I generated by $\boldsymbol{f}(\boldsymbol{\psi})$ which is denoted by $\langle \boldsymbol{f}(\boldsymbol{\psi})\rangle$ is decomposed as

$$I = I_1 \cap \cdots \cap I_r, \tag{6}$$

the variety $\boldsymbol{V}[\boldsymbol{f}(\boldsymbol{\psi})]$ which is denoted by $\boldsymbol{V}(I)$ is decomposed as

$$\boldsymbol{V}(I) = \boldsymbol{V}(I_1) \cup \cdots \cup \boldsymbol{V}(I_r). \tag{7}$$

The previous research used the Gröbner base of lexicographic order for the ideal decomposition [6, 7]. However, the requirement of the tremendously huge computational cost for the lexicographic order Gröbner base prevents the ideal decomposition of Eq.(6). To overcome the difficulty, we use the ideal quotient for the ideal decomposition. When the ideal I is represented by $I = J \cap K$ and the ideal K is a given ideal, the ideal J is calculated by the ideal quotient

$$J = I : K. \tag{8}$$

In order to obtain the ideal K, we can use the symmetry of nonlinear systems.

3.2. Ideal Quotient based on Symmetry

When we consider a finite group Γ, the symmetry is obtained by a linear representation $\boldsymbol{\theta} : \Gamma \to GL(\boldsymbol{R}^n)$, where $GL(\boldsymbol{R}^n)$ is a set of invertible $\boldsymbol{R}^{n\times n}$ matrices. When the system is the following relation

$$\boldsymbol{f}(\boldsymbol{\theta}(\gamma)\boldsymbol{\psi}) = \boldsymbol{\theta}(\gamma)\boldsymbol{f}(\boldsymbol{\psi}), \quad \gamma \in \Gamma, \tag{9}$$

if $\boldsymbol{\psi}$ is a solution, $\boldsymbol{\theta}(\gamma)\boldsymbol{\psi}$ is also a solution. As the symmetric solutions satisfy $\boldsymbol{\psi} = \boldsymbol{\theta}(\gamma)\boldsymbol{\psi}$ for all $\gamma \in \Gamma$, the symmetric solutions are constrained by the sum of ideals

$$\sum_{\gamma\in\Gamma} \langle \boldsymbol{\psi} - \boldsymbol{\theta}(\gamma)\boldsymbol{\psi}\rangle \equiv \left\{\sum_{\gamma\in\Gamma} f_\gamma \mid f_\gamma \in \langle \boldsymbol{\psi} - \boldsymbol{\theta}(\gamma)\boldsymbol{\psi}\rangle\right\}. \tag{10}$$

The symmetric solutions are determined by $\boldsymbol{f}(\boldsymbol{\psi}) = \boldsymbol{0}$ and the constraint of $\boldsymbol{\psi} - \boldsymbol{\theta}(\gamma)\boldsymbol{\psi} = \boldsymbol{0}$. Thus, the ideal which represents the symmetric solutions is written by

$$\langle \boldsymbol{f}_\Gamma(\boldsymbol{\psi})\rangle \equiv \langle \boldsymbol{f}(\boldsymbol{\psi})\rangle + \sum_{\gamma\in\Gamma} \langle \boldsymbol{\psi} - \boldsymbol{\theta}(\gamma)\boldsymbol{\psi}\rangle. \tag{11}$$

Because we consider $\langle \boldsymbol{f}(\boldsymbol{\psi})\rangle \subset \langle \boldsymbol{f}_\Gamma(\boldsymbol{\psi})\rangle$, the ideal $\langle \overline{\boldsymbol{f}_\Gamma}(\boldsymbol{\psi})\rangle$ which represents the asymmetric solutions is provided by ideal quotient as follows:

$$\langle \overline{\boldsymbol{f}_\Gamma}(\boldsymbol{\psi})\rangle = \langle \boldsymbol{f}(\boldsymbol{\psi})\rangle : \langle \boldsymbol{f}_\Gamma(\boldsymbol{\psi})\rangle. \tag{12}$$

Therefore, we can decompose the bifurcation diagram using the ideal quotient based on the symmetry of systems.

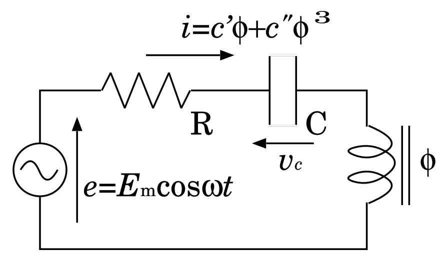

Figure 1: RLC resonance circuit.

4. Example of Decomposition

4.1. Circuit Equation

We apply the proposed method to an RLC resonance circuit shown in Figure 1. We assume that the magnetizing characteristics of the nonlinear inductor is approximated by cubic polynomial, i.e., $i = c'\phi + c''\phi^3$, where ϕ is a magnetic flux and i is current. The scaled circuit equation is

$$\frac{\mathrm{d}\boldsymbol{u}(\tau)}{\mathrm{d}\tau} = \begin{bmatrix} -(c_1u_1 + c_3{u_1}^3) - u_2 \\ X(c_1u_1 + c_3{u_1}^3) \end{bmatrix} + \begin{bmatrix} \frac{1}{m}E\cos m\tau \\ 0 \end{bmatrix}, \tag{13}$$
$$\boldsymbol{u}(\tau) = (u_1, u_2)',\ u_1 = \frac{\omega\phi}{m\sqrt{2}I_{\mathrm{n}}R},\ u_2 = \frac{v_c}{\sqrt{2}I_{\mathrm{n}}R},$$
$$m\tau = \omega t,\ c_1 = 1 - c_3 = \frac{mR}{\omega}c',$$
$$c_3 = \frac{2m^3{E_{\mathrm{n}}}^2R^3}{{R_{\mathrm{n}}}^2\omega^3}c'',\ X = \frac{m}{\omega RC},\ E = \frac{mE_{\mathrm{m}}}{\sqrt{2}I_{\mathrm{n}}R},$$

where E_{n}, I_{n} and R_{n} are values for the normalization. We apply the harmonic balance method with 3 frequency components, i.e., $\kappa = \{1, 2, 3\}$. The $u_1(\tau)$ and $u_2(\tau)$ are respectively approximated by

$$\begin{aligned} u_1^*(\tau) = {}& \psi_{c1}\cos\tau + \psi_{s1}\sin\tau + \psi_{c2}\cos 2\tau + \psi_{s2}\sin 2\tau \\ & +\psi_{c3}\cos 3\tau + \psi_{s3}\sin 3\tau, \end{aligned} \tag{14}$$
$$u_2^*(\tau) = K^*\left[Xc_1u_1^* + Xc_3{u_1^*}^3\right]. \tag{15}$$

We deal with the case of $m = 1$, i.e., we consider the fundamental oscillation and the 2nd, 3rd harmonic oscillations for the harmonic balance. From the equations, we are able to obtain the simultaneous algebraic equations $\boldsymbol{f}(\boldsymbol{\psi}) \equiv (f_{c1}, f_{s1}, f_{c2}, f_{s2}, f_{c3}, f_{s3})' = \boldsymbol{0}$, where $\boldsymbol{\psi} \equiv$

$(\psi_{c1}, \psi_{s1}, \psi_{c2}, \psi_{s2}, \psi_{c3}, \psi_{s3})'$ and the $f_{c1}, f_{s1}, f_{c2}, f_{s2}, f_{c3}, f_{s3}$ are equations by the harmonic balance method. In this example, we cannot obtain the Gröber base of lexicographic order by using computer algebra system Risa/Asir [8] because the computation requires very huge memory.

4.2. Symmetry of Target Oscillations

In order to decompose the bifurcation diagram using ideal quotient, let us consider the symmetry of the target oscillations. Based on the symmetry of $e(t+\pi) = -e(t)$ which is denoted by $\Gamma_{-\pi}$, the linear representation $\boldsymbol{\theta_{-\pi}}$ is represented by

$$\boldsymbol{\theta_{-\pi}} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & \mathbf{0} & \\ & & -1 & & & \\ & & & -1 & & \\ & \mathbf{0} & & & 1 & \\ & & & & & 1 \end{bmatrix}. \quad (16)$$

Because the symmetric solutions satisfy $\boldsymbol{\theta_{-\pi}\psi} = \boldsymbol{\psi}$, the symmetric solutions have to satisfy $\psi_{c2} = \psi_{s2} = 0$. As a result, we obtain an ideal

$$I_{13} \equiv I + \langle \psi_{c2}, \psi_{s2} \rangle = \langle \boldsymbol{f}(\psi_{c1}, \psi_{s1}, 0, 0, \psi_{c3}, \psi_{s3}) \rangle, \quad (17)$$

based on the symmetry. The ideal I_{13} corresponds to the oscillations which have only fundamental and 3rd harmonic frequency components. Using the ideal $I \equiv \langle \boldsymbol{f}(\boldsymbol{\psi}) \rangle$ and ideal quotient by I_{13}, we can decompose the ideal as

$$I = I_{13} \cap I_{123}, \quad (18)$$

$$I_{123} \equiv I : I_{13} = \langle \boldsymbol{f}(\boldsymbol{\psi}) \rangle : \langle \boldsymbol{f}(\psi_{c1}, \psi_{s1}, 0, 0, \psi_{c3}, \psi_{s3}) \rangle. \quad (19)$$

The ideal I_{123} corresponds to the oscillations which have all frequency components. However, the computation of the ideal quotient (19) requires more than 4GB memory because the calculation of ideal quotient also contains the calculation of Gröbner base [1], and we cannot obtain the ideal decomposition yet.

5. Efficient Methods for Ideal Quotient

In order to calculate the ideal quotient (19), we propose two efficient methods. First, assuming that an ideal J is represented by $J = \sum_{i=1}^{s} J_i$, the ideal quotient $I : J$ is written by

$$I : J = I : \left(\sum_{i=1}^{s} J_i \right) = \bigcap_{i=1}^{s} (I : J_i). \quad (20)$$

Using the relation (20), the ideal I_{123} is represented by

$$I_{123} = \langle \boldsymbol{f}(\boldsymbol{\psi}) \rangle : \langle \psi_{c2}, \psi_{s2} \rangle. \quad (21)$$

Second, we introduce the partial ideal quotient. When the ideal I is represented by $I = \sum_{i=1}^{r} I_i$, the ideal quotient $I : J$ is obtained by

$$I : J = \left(\sum_{i=1}^{r} I_i \right) : J = \sum_{i=1}^{r} (I_i : J), \quad (22)$$

This relation indicates that the computation of ideal quotient $I : J$ is equivalent to the sum of the partial ideal quotients $I_i : J$.

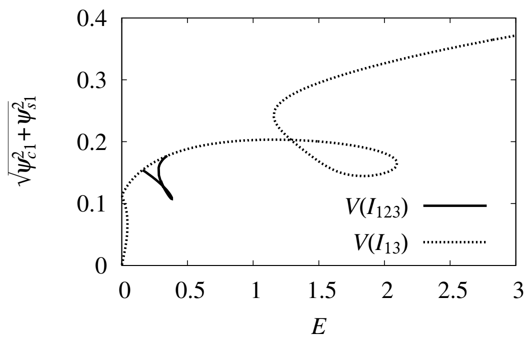

Figure 2: Bifurcation diagram, the parameter E versus the amplitude of the fundamental oscillation $\sqrt{\psi_{c1}^2 + \psi_{s1}^2}$. (The ideal I_{123} corresponds to the oscillations which have all frequency components, and the ideal I_{13} corresponds to the oscillations which have only fundamental and 3rd harmonic frequency components. $c_3 = 1.0,\ X = 100.0$.)

Applying the relation (22) to Eq.(21), the ideal I_{123} is represented by

$$I_{123} = \langle f_{c2}, f_{s2} \rangle : \langle \psi_{c2}, \psi_{s2} \rangle + \langle f_{c1}, f_{s1}, f_{c3}, f_{s3} \rangle, \quad (23)$$

where we use the relation $\langle f_{c1}, f_{s1}, f_{c3}, f_{s3} \rangle \not\subset \langle \psi_{c2}, \psi_{s2} \rangle$ which is easily confirmed by the substitution of $\psi_{c2} = \psi_{s2} = 0$ into $f_{c1}, f_{s1}, f_{c3}, f_{s3}$.

Using Eq.(23), we can obtain the ideal decomposition (18) without memory overflow. After the ideal decomposition, we obtain the bifurcation diagram using Gröbner base of block order and try further decomposition by factorizations. In this case we cannot decompose the ideal anymore. The fact shows that the oscillation does not have more symmetries.

The decomposition of bifurcation diagram is shown in Figure 2 which represents the relation between the parameter E and the amplitude of the fundamental oscillation $\sqrt{\psi_{c1}^2 + \psi_{s1}^2}$ where $c_3 = 1.0,\ X = 100.0$. The solid line corresponds to the ideal I_{123} and the dotted line corresponds to the ideal I_{13}. We can confirm that the intersections of the decomposed diagrams corresponds to pitchfork bifurcations.

6. Decomposition of Homogeneous Equation

In the previous section, we consider the case that the frequency m of the source is equal to 1 in Eq.(13). Let us consider also the case of $m = 2$ and $m = 3$. The cases of $m = 2$ and $m = 3$ correspond to 1/2-subharmonic oscillations and 1/3-subharmonic oscillations, respectively. In order to clarify the effect of the frequency m on the harmonic balance equations, we define a homogeneous equation $\hat{\boldsymbol{f}}(\boldsymbol{\psi})$ by

$$\hat{\boldsymbol{f}}(\boldsymbol{\psi}) = \hat{\boldsymbol{f}}(\boldsymbol{\psi} \mid \boldsymbol{\lambda}) \equiv \begin{bmatrix} j\psi_{sj,i} - p_{j,i}(\boldsymbol{\psi} \mid \boldsymbol{\lambda}) \\ -j\psi_{cj,i} - q_{j,i}(\boldsymbol{\psi} \mid \boldsymbol{\lambda}) \end{bmatrix} = \boldsymbol{0}, \quad (24)$$

$$\boldsymbol{f}(\boldsymbol{\psi}) = \hat{\boldsymbol{f}}(\boldsymbol{\psi} \mid \boldsymbol{\lambda}) + \hat{\boldsymbol{e}} = \boldsymbol{0}. \quad (25)$$

The equations $\boldsymbol{f}(\boldsymbol{\psi})$ of the harmonic balance for $m = 1, 2$ and 3 are represented as

$$\boldsymbol{f}(\boldsymbol{\psi}) = (\hat{f}_{c1}, \hat{f}_{s1} + E, \hat{f}_{c2}, \hat{f}_{s2}, \hat{f}_{c3}, \hat{f}_{s3})', \quad (26)$$

$$f(\psi) = (\hat{f}_{c1}, \hat{f}_{s1}, \hat{f}_{c2}, \hat{f}_{s2} + E, \hat{f}_{c3}, \hat{f}_{s3})', \quad (27)$$

$$f(\psi) = (\hat{f}_{c1}, \hat{f}_{s1}, \hat{f}_{c2}, \hat{f}_{s2}, \hat{f}_{c3}, \hat{f}_{s3} + E)', \quad (28)$$

where $\hat{f}(\psi) = (\hat{f}_{c1}, \hat{f}_{s1}, \hat{f}_{c2}, \hat{f}_{s2}, \hat{f}_{c3}, \hat{f}_{s3})'$. Thus, the source E of frequency m effects only the constant term of corresponding frequencies. The fact indicates that the homogeneous equation $\hat{f}(\psi)$ has the highest symmetry and that each harmonic balance equations is obtained by breaking of the symmetry.

In order to clarify the symmetries of the harmonic balance equations systematically, we consider the symmetries of the homogeneous equation $\hat{f}(\psi)$. In addition to the symmetry $\Gamma_{-\pi}$, the homogeneous equation has $\Gamma_{2\pi/m}$ which is defined by $e(\tau) = e(\tau + 2\pi/m)$ for $m = 1, 2, 3$. The corresponding linear representations for $m = 2$ and 3 are

$$\theta_{2\pi/2} = \begin{bmatrix} -1 & & & & & \\ & -1 & & & 0 & \\ & & 1 & & & \\ & & & 1 & & \\ & 0 & & & -1 & \\ & & & & & -1 \end{bmatrix}, \quad (29)$$

$$\theta_{2\pi/3} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & & & & \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & & & 0 & \\ & & \frac{1}{2} & -\frac{\sqrt{3}}{2} & & \\ & & \frac{\sqrt{3}}{2} & \frac{1}{2} & & \\ & 0 & & & 1 & \\ & & & & & 1 \end{bmatrix}. \quad (30)$$

The symmetries $\Gamma_{-\pi}$, $\Gamma_{2\pi/2}$ and $\Gamma_{2\pi/3}$ generate respectively the partial ideal quotients

$$\langle \hat{f}_{c2}, \hat{f}_{s2} \rangle : \langle \psi_{c2}, \psi_{s2} \rangle, \quad (31)$$

$$\langle \hat{f}_{c1}, \hat{f}_{s1}, \hat{f}_{c3}, \hat{f}_{s3} \rangle : \langle \psi_{c1}, \psi_{s1}, \psi_{c3}, \psi_{s3} \rangle, \quad (32)$$

$$\langle \hat{f}_{c1}, \hat{f}_{s1}, \hat{f}_{c2}, \hat{f}_{c2} \rangle : \langle \psi_{c1}, \psi_{s1}, \psi_{c2}, \psi_{s2} \rangle. \quad (33)$$

By knowing the symmetries of homogeneous equations in advances, we can obtain the ideal decomposition efficiently.

In the case of $m = 1$, the symmetries $\Gamma_{2\pi/2}$ and $\Gamma_{2\pi/3}$ are broken because Eq.(32) and Eq(33) contain $\hat{f}_{c1}$ and $\hat{f}_{s1}$. As a result, we use the symmetry $\Gamma_{-\pi}$ and obtain the ideal decomposition as follows:

$$I = I_{13} \cap I_{123}, \quad (34)$$

$$I_{13} = \langle f(\psi_{c1}, \psi_{s1}, 0, 0, \psi_{c3}, \psi_{s3}) \rangle, \quad (35)$$

$$I_{123} = \langle \hat{f}_{c2}, \hat{f}_{s2} \rangle : \langle \psi_{c2}, \psi_{s2} \rangle + \langle \hat{f}_{c1}, \hat{f}_{s1} + E, \hat{f}_{c3}, \hat{f}_{s3} \rangle. \quad (36)$$

In the case of $m = 2$, the symmetries $\Gamma_{-\pi}$ and $\Gamma_{2\pi/3}$ are broken because Eq.(31) and Eq.(33) contain $\hat{f}_{c2}$ and $\hat{f}_{s2}$. As a result, we use the symmetry $\Gamma_{2\pi/2}$ and obtain the ideal decomposition as follows:

$$I = I_2 \cap I_{123}, \quad (37)$$

$$I_2 = \langle f(0, 0, \psi_{c2}, \psi_{s2}, 0, 0) \rangle, \quad (38)$$

$$I_{123} = \langle \hat{f}_{c1}, \hat{f}_{s1}, \hat{f}_{c3}, \hat{f}_{s3} \rangle : \langle \psi_{c1}, \psi_{s1}, \psi_{c3}, \psi_{s3} \rangle + \langle \hat{f}_{c2}, \hat{f}_{s2} + E \rangle. \quad (39)$$

In the case of $m = 3$, only the symmetry $\Gamma_{2\pi/2}$ is broken because only Eq.(32) contains $\hat{f}_{c3}$ and $\hat{f}_{s3}$. As a result, we use the both symmetry $\Gamma_{-\pi}$ and $\Gamma_{2\pi/3}$, and obtain the ideal decomposition as follows:

$$I = I_3 \cap I_{13} \cap I_{123}, \quad (40)$$

$$I_3 = \langle f(0, 0, 0, 0, \psi_{c3}, \psi_{s3}) \rangle, \quad (41)$$

$$I_{13} = \langle \hat{f}_{c1}, \hat{f}_{s1} \rangle : \langle \psi_{c1}, \psi_{s1} \rangle + \langle \hat{f}_{c3}, \hat{f}_{s3} + E \rangle, \quad (42)$$

$$I_{123} = \langle \hat{f}_{c2}, \hat{f}_{s2} \rangle : \langle \psi_{c2}, \psi_{s2} \rangle + \langle \hat{f}_{c1}, \hat{f}_{s1}, \hat{f}_{c3}, \hat{f}_{s3} + E \rangle. \quad (43)$$

We applied the derived symmetry to the ideal decomposition and confirmed that the ideal decomposition is successfully obtained. Further, we confirm that any other symmetries do not exist in the oscillation by computing the further ideal decomposition. Thus, we can obtain the complete mode decomposition of the bifurcation diagram.

7. Conclusion

This paper proposed a method for decomposing the bifurcation diagram of periodic oscillations using the ideal decomposition. In order to realize the ideal decomposition, we used the ideal quotient based on the symmetry of the target system and make it more efficient by using the partial ideal quotient. Further, we derived the symmetries of homogeneous equations and clarified the symmetries of the equations obtained by the harmonic balance method.

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