2005 International Symposium on Nonlinear
Theory and its Applications (NOLTA2005)
Bruges, Belgium, October 18-21, 2005

Intermodulation Analysis of Nonlinear Circuits Using Two-Dimensional Fourier Transformation

A.Ushida†, *J.Kawata*†, *Y.Yamagami*‡, *Y.Nishio*‡
†Department of Mechanical and Electronic Engineering,
Tokushima Bunri University, Kagawa, 769-2193 JAPAN
‡Department of Electrical Electronic Engineering,
Tokushima University, Tokushima, 770-8506 JAPAN

Abstract

Distortion analysis of nonlinear circuits is very important for designing analog integrated circuits and communication systems. In this paper, we propose an efficient frequency domain algorithm for analyzing the intermodulations of mixers and modulators driven by multiple inputs. Firstly, using ABMs(Analog Behavior Models) of Spice, we have developed a Fourier transformer executing the two-dimensional Fourier transformation. Then, the *device modules* for nonlinear elements such as diodes, bipolar transistors and MOSFETs are modeled by the Fourier transformer. Applying these modules, we can formulate the determining equations of the harmonic balance method in the form of equivalent circuits. Thus, they can be solved by the DC analysis of Spice, so that we can easily obtain the characteristics such as the frequency response curves and intermodulation phenomena. In our algorithm, we need not derive any troublesome circuit equation and the transformations into the determining equations, so that our simulators are user-friendly for solving these circuits.

1. Introduction

The frequency response curves of nonlinear electronic circuits are very useful for the investigation of the global circuit behaviors. The Volterra series methods have been widely used for these purposes[1-5]. The kernel functions are given in the analytical forms, so that we can easily understand the circuit behaviors from the functions. Although the algorithms are theoretically elegant, it is not so easy to derive the higher order Volterra kernels, especially, for the large scale systems containing many nonlinear elements [1]. Remark that since the ideas are based on the bilinear theorem, they can be only applied to the weakly nonlinear circuits and the convergences are not guaranteed for strong nonlinear circuits. Furthermore, the characteristics of nonlinear devices in the Volterra series should be approximated by the polynomial forms, which can be done by the Taylor expansions in the vicinity at each DC operating point of nonlinear devices [2]. These tasks are not easy to the complicated device models such as the Gummel-Poon model of bipolar transistors and the higher level models of MOSFETs, especially, in the high frequency domain[6,7]. On the other hand, many algorithms have been proposed for calculating the exact steady-state waveforms in the time-domain [8-10]. Unfortunately, they are not efficient for analyzing the characteristics such as frequency response curves.

In this paper, we propose a new Spice-oriented harmonic balance method for solving the intermodulation phenomena of the nonlinear circuits. Generally, mixers and modulator circuits are driven by multiple inputs, so that the steady-state waveforms behave as quasi-periodic functions. In this case, the harmonic balance method needs to apply *multi-dimensional Fourier transformation* [14] [1] . On the other hand, ICs consist of many kinds of nonlinear devices such as diodes, bipolar transistors and MOSFETs, whose device models are described by special functions and/or piecewise continuous functions. Then, using ABMs of Spice [12], we have developed the Fourier transformers which can be applied to any kinds of models, and constructed the Fourier transfer modules for these devices, where the relations between the terminal voltages and currents are described by functions of the Fourier coefficients. We call the modules "*packaged device modules*" . Thus, in our harmonic balance method, all the nonlinear devices in the circuits are replaced by the corresponding packaged device modules, and the linear sub-circuits are transformed into the corresponding Cosine and Sine circuits [11] composed of the linear resistive elements and controlled sources. Remark that the equivalent circuit obtained in the above corresponds to the *determining equations* of the harmonic balance method. We will simply call the circuit *Fourier transfer circuit*, which can be solved by the DC analysis of Spice, and we obtain the characteristic curves such as frequency response curves.

Thus, once these packaged device modules for all kinds of nonlinear devices are installed in our computers as library, the harmonic balance method can be easily applied to any electronic circuits. In this way, we have realized user-friendly simulators for calculating the intermodulation analysis.

We show the Fourier transfer circuit in section 2, and a technique for getting the packaged device modules in section 3. The interesting illustrative examples are shown in section 4.

2. Fourier transfer circuit model

Analog integrated circuits are usually composed of many kinds of nonlinear devices such as diodes, bipolar transistors and MOSFETs, whose Spice models are described by the special functions containing the exponential, square-root,

[1] If the circuit is driven by N independent frequency components, we need to apply an N-dimensional Fourier transformation that is really time-consuming for large N.

piecewise continuous functions and so on [7]. For these devices, the Fourier coefficients cannot be described in the analytical forms. Therefore, using ABMs of Spice [12], we realize the equivalent circuits which execute the Fourier expansions in the harmonic balance methods. Now, consider a nonlinear function described by

$$i = \hat{g}(v) \tag{1}$$

Assume the variable v is consisted of the combination tones of ω_1, ω_2, and is described by two variables τ_1 and τ_2; namely,

$$\begin{aligned} v(\tau_1,\tau_2) = V_0 + \sum_{k=1}^{M} \{ & V_{2k-1}\cos(m_{1k}\omega_1\tau_1 + m_{2k}\omega_2\tau_2) \\ & + V_{2k}\sin(m_{1k}\omega_1\tau_1 + m_{2k}\omega_2\tau_2)\}, \text{ for } |m_{1k}| < N, \quad |m_{2k}| < K \end{aligned} \tag{2}$$

for a given bounded N and K. Let us set

$$t = \tau_1 = \tau_2 \tag{3}$$

in (2). Then, it is reduced to

$$\begin{aligned} v(t) = V_0 + \sum_{k=1}^{M} \{ & V_{2k-1}\cos(m_{1k}\omega_1 + m_{2k}\omega_2)t \\ & + V_{2k}\sin(m_{1k}\omega_1 + m_{2k}\omega_2)t\} \end{aligned} \tag{4}$$

Hence, the Fourier expansion of (1) for the input (4) can be obtained by the use of the response for the input (2) instead of (4). The output response $i(\tau_1,\tau_2)$ is given by

$$i(\tau_1,\tau_2) = \alpha_0(\tau_1) + \sum_{k=1}^{K}\{\alpha_{2k-1}(\tau_1)\cos k\omega_2\tau_2 + \alpha_{2k}(\tau_1)\sin k\omega_2\tau_2\} \tag{5}$$

where $\alpha_0(\tau_1), \alpha_1(\tau_1), \ldots, \alpha_{2K}(\tau_1)$ are periodic functions with the period $T_1 = 2\pi/\omega_1$, and K is the highest harmonic component of τ_2. Here, if we execute the Fourier transformation of (5) for fixed τ_1 at $\omega_1\tau_1 = \{0, \Delta\theta_1, 2\Delta\theta_1, \ldots, 2\pi\}$ for $\Delta\theta_1 = \pi/N$, then we can obtain $\alpha_k(0)$,$\alpha_k(\Delta\theta_1)$,$\alpha_k(2\Delta\theta_1)$,$\ldots$, $\alpha_k(2\pi)$. Thus, $\alpha_k(\tau_1)$ can be also expanded into a Fourier series;

$$\begin{aligned} & \alpha_k(\tau_1) = \alpha_{k,0} + \sum_{n=1}^{N}\{\alpha_{n,2k-1}\cos n\omega_1\tau_1 + \alpha_{k,2n}\sin n\omega_1\tau_1\} \\ & k = 0, 1, 2, \ldots, 2K \end{aligned} \tag{6}$$

Substituting (6) into (5), we obtain

$$\begin{aligned} i(\tau_1,\tau_2) = & \alpha_{0,0} + \sum_{k=n}^{N}\{\alpha_{0,2k-1}\cos n\omega_1\tau_1 + \alpha_{0,2n}\sin n\omega_1\tau_1\} \\ & + \sum_{k=1}^{K}\{\alpha_{2k-1,0} + \sum_{n=1}^{N}[\alpha_{2k-1,2n-1}\cos n\omega_1\tau_1 + \alpha_{2k-1,2n}\sin n\omega_1\tau_1]\} \\ & \quad \times \cos k\omega_2\tau_2 \\ & + \sum_{k=1}^{K}\{\alpha_{2k,0} + \sum_{n=1}^{N}[\alpha_{2k,2n-1}\cos n\omega_1\tau_1 + \alpha_{2k,2n}\sin n\omega_1\tau_1]\} \\ & \quad \times \sin k\omega_2\tau_2 \end{aligned} \tag{7}$$

Now, setting $t = \tau_1 = \tau_2$ in (7) and applying some trigonometric arithmetics, we obtain it in the form of

$$\begin{aligned} i(t) = I_0 + \sum_{k=1}^{M}\{ & I_{2k-1}\cos(m_{1k}\omega_1 + m_{2k}\omega_2)t \\ & + I_{2k}\sin(m_{1k}\omega_1 + m_{2k}\omega_2)t\} \end{aligned} \tag{8}$$

Observe that the above two-dimensional Fourier expansion takes at least $(2N+1)$-times Fourier expansions for ω_1-component, and $(2K+1)$-times expansions for ω_2-component[2]. Thus, the total number of the Fourier expansions is at least $(2N+1)\times(2K+1)$.

In our algorithm, we will apply a well-known discrete Fourier transform for each Fourier expansion as follows;

$$\left.\begin{aligned} & I_0 = \frac{1}{2\pi}\int_0^{2\pi}\hat{g}(v)dt \\ & I_{2k_1} = \frac{1}{\pi}\int_0^{2\pi}\hat{g}(v)\cos k\omega t dt, I_{2k} = \frac{1}{\pi}\int_0^{2\pi}\hat{g}(v)\sin k\omega t dt \\ & k = 1, 2, \ldots, N, \text{ or } K \end{aligned}\right\} \tag{9}$$

Now, let us apply a trapezoidal integration formula to (9) as follows;

$$\int_a^b \hat{g}(v)dt = \frac{h}{2}(\hat{g}_0 + \hat{g}_n) + h(\hat{g}_1 + \hat{g}_2 + \cdots + \hat{g}_{n-1}) \tag{10}$$

where the step size of the integration is $h = (a-b)/n$. Then, the truncation error is given by $\hat{g}^{(2)}h^2/12n$, where $^{(2)}$ shows the second derivative. Replacing the integrations in (9) by (10), we can realize the equivalent circuit model satisfying the relations (9). To understand the circuit model, we assume the input

$$v(\theta) = V_0 + \sum_{k=1}^{M}(V_{2k-1}\cos k\theta + V_{2k}\sin k\theta), \quad \theta = \omega t \tag{11}$$

The Fourier transfer circuit model for calculating the Nth higher harmonic component is shown in Fig.1. Applying integral formula (10) to (9), we have

$$\left.\begin{aligned} & I_{2N-1} = \frac{1}{\pi}\int_0^{2\pi}\hat{g}(v)\cos N\theta d\theta = \frac{1}{K}(\hat{g}_0 + \hat{g}_{2K}) + \frac{2}{K}(\hat{g}_1\cos N\theta_1 \\ & + \hat{g}_2\cos N\theta_2 + \cdots + \hat{g}_{2K-1}\cos N\theta_{2K-1}) \\ & I_{2N} = \frac{1}{\pi}\int_0^{2\pi}\hat{g}(v)\sin N\theta d\theta = \frac{2}{K}(\hat{g}_1\sin N\theta_1 + \hat{g}_2\sin N\theta_2 \\ & + \cdots + \hat{g}_{2K-1}\sin N\theta_{2K-1}) \end{aligned}\right\} \tag{12}$$

The blocks in Fig.1 are constructed by the ABMs of Spice which calculates the each term of the relation (12), where the interval $[0, 2\pi]$ of the integration is divided by $2K$ equal divisions. In Fig. 1, the value of $\theta_k = 2\pi/2K$ is obtained by the node voltage at the kth resistor in the resistive circuit.

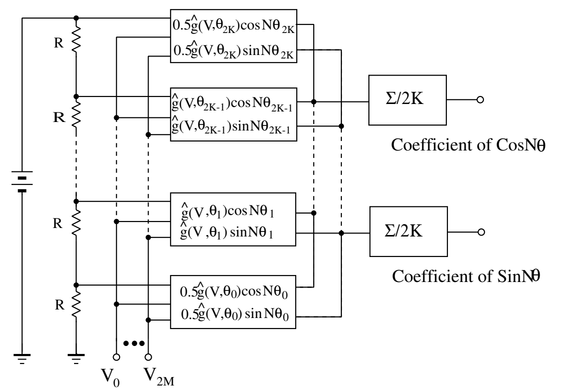

Fig.1 Fourier transfer circuit model.

To investigate the accuracy of our Fourier transfer circuit model, we calculate the following Fourier expansion:

$$e^{x\cos\theta} = I_0(x) + I_1(x)\cos\theta + I_2(x)\cos 2\theta + \cdots \tag{13}$$

[2]The least number comes from the sampling theorem. For getting exact solution, we need the more number of Fourier expansions.

whose Fourier coefficients are given by modified Bessel functions [13] as follows:

$$I_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos\theta}\cos N\theta d\theta, \quad N = 0, 1, 2, \ldots \tag{14}$$

The simulation result for $h = 2\pi/20$ in Fig.1 is $I_1(10) = 2761$ at $N = 1,\ x = 10$ which is exactly equal to the value from the table of Bessel function [13]. Thus, we found that the Fourier transfer circuit model can get the sufficiently exact solution even with $2K = 10$ to 20 divisions of the interval 2π.

Next, we consider the two-dimensional Fourier transformation for the input waveform (4). The algorithm is as follows;

Two-Dimensional Fourier Transformation

Step 1 Set τ_1 and τ_2 as follows;

$$\left.\begin{array}{l} \omega_1\tau_1 = 0, \Delta\theta_1, 2\Delta\theta_1, \ldots, (2N-1)\Delta\theta_1, 2\pi, \\ \qquad \text{for } \Delta\theta_1 = \pi/N \\ \omega_2\tau_2 = 0, \Delta\theta_2, 2\Delta\theta_2, \ldots, (2K-1)\Delta\theta_2, 2\pi, \\ \qquad \text{for } \Delta\theta_2 = \pi/K \end{array}\right\} \tag{15}$$

Step 2 Calculate $i(\omega_1\tau_1, \omega_2\tau_2)$ at the points $\{i(n\Delta\theta_1, k\Delta\theta_2), n = 0, 1, \ldots, 2N, k = 0, 1, \ldots, 2K\}$.

Step 3 For fixed $\omega_1\tau_1 = n\Delta\theta_1, n = 0, 1, \ldots, 2N$, execute the Fourier transformation for θ_2. Thus, we get $\{\alpha_k(n\Delta\theta_1), k = 0, 1, 2 \ldots, 2K\}$ from the relations (5).

Step 4 Next, for fixed k, $\alpha_k(\tau_1)$ is expanded into the Fourier series using the values $\{\alpha_k(n\Delta\theta_1), n = 0, 1, 2, \ldots, 2N\}$. Thus, we have the relations (6) and (7).

Step 5 Finally, from some trigonometric arithmetics, we obtain the two dimensional Fourier expansion (9).

The flowchart for a three terminal element such as bipolar transistors and MOSFETs is shown in Fig. 2, where the inputs are given as the Fourier coefficients of 3 terminal voltages, and the corresponding output current coefficients are obtained from two terminals.

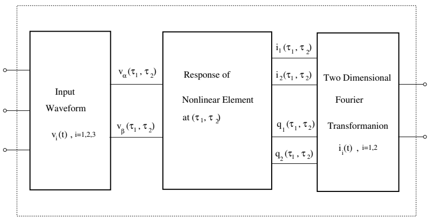

Fig.2 Flowchart of the two-dimensional Fourier transformation for a three terminal element.

3. Fourier transformation of nonlinear devices

Analog ICs contain many kind of nonlinear devices such as diodes, bipolar transistors and MOSFETs. In this section, we show a technique to construct the *packaged device modules* with the Fourier transfer circuit given in section 2.These customizations are very useful to develop user-friendly simulators in our harmonic balance method. Now, we consider npn bipolar transistor as shown in Fig. 3. The DC currents with Gummel-Poon model is given as follows[6-7];

$$i_C = \frac{I_S}{Q_B}\left(\exp\left(\frac{v_{BE}}{V_t}\right) - 1\right) - I_S\left(\frac{1}{Q_B} + \frac{1}{B_R}\right)\left(\exp\left(\frac{v_{BC}}{V_t}\right) - 1\right) \tag{16.1}$$

$$i_B = \frac{I_S}{B_F}\left(\exp\left(\frac{v_{BE}}{V_t}\right) - 1\right) + \frac{I_S}{B_R}\left(\exp\left(\frac{v_{BC}}{V_t}\right) - 1\right) \tag{16.2}$$

where

$$\frac{1}{Q_B} \simeq 1 - \frac{v_{BC}}{V_{AF}} - \frac{v_{BE}}{V_{AF}}$$

On the other hand, the charge q_{BE} and q_{BC} are given by

$$q_i = \left(\frac{C_{j0}A_n}{[1-(v_i/\phi_B)]^m} + \frac{\tau I_S A_n}{nV_t}\exp\left(\frac{v_i}{nV_t}\right)\right)v_i, \ \text{for } (v_i < F_c\phi_B) \tag{17.1}$$

$$q_i = \left(\frac{C_{j0}A_n}{[1-F_c]^{(1+m)}}\left(1 - F_c(1+m) + \frac{mv_i}{\phi_B}\right) + \frac{\tau I_S A_n}{nV_t}\exp\left(\frac{v_i}{nV_t}\right)\right)v_i, \ \text{for } (v_i \geq F_c\phi_B) \tag{17.2}$$

$$i = BC, BE$$

Now, we assume the input voltage waveforms at the collector, base and emitter v_C, v_B, v_E as follows;

$$v_i(t) = V_{i,0} + \sum_{k=1}^{M}\{V_{i,2k-1}\cos(m_{1k}\omega_1 + m_{2k}\omega_2)t + V_{i,2k}\sin(m_{1k}\omega_1 + m_{2k}\omega_2)t\}, \quad i = C, B, E, \tag{18}$$

Firstly, using two-dimensional Fourier expansion given in section 2, we execute the Fourier expansion to the base and corrector currents $\{i'_B(t), i'_C(t)\}$ given by (16), and the base-collector and base-emitter charges $\{q_{BC}(t), q_{BE}(t)\}$ given by (17). After then, we use the following relations;

$$\left.\begin{array}{l} i_B(t) = i'_B(t) + \dfrac{dq_{BE}(t)}{dt} + \dfrac{dq_{BC}(t)}{dt} \\ i_C(t) = i'_C(t) - \dfrac{dq_{BC}(t)}{dt} \end{array}\right\} \tag{19}$$

Thus, we have the two-dimensional Fourier expansions of the base and collector currents as follows;

$$i_i(t) = I_{i,0} + \sum_{k=1}^{M}\{I_{i,2k-1}\cos(m_{1k}\omega_1 + m_{2k}\omega_2)t + I_{i,2k}\sin(m_{1k}\omega_1 + m_{2k}\omega_2)t\}, \quad i = C, B \tag{20}$$

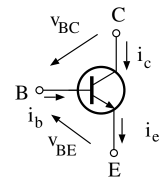

Fig.3 Nonlinear electronic circuit.

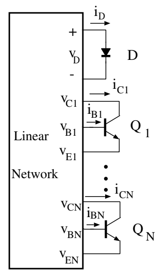

Fig.4 Nonlinear electronic circuit.

where

$I_{B,0} = I'_{B,0}$,

$I_{B,2k-1} = I'_{B,2k-1} + (m_{1k}\omega_1 + m_{2k}\omega_2)(Q_{BE,2k-1} + Q_{BC,2k-1})$

$I_{B,2k} = I'_{B,2k} - (m_{1k}\omega_1 + m_{2k}\omega_2)(Q_{BE,2k} + Q_{BC,2k}) \quad (21.1)$

$I_{C,0} = I'_{C,0}$,

$I_{C,2k-1} = I'_{C,2k-1} - (m_{1k}\omega_1 + m_{2k}\omega_2)Q_{BC,2k-1}$

$I_{C,2k} = I'_{C,2k} + (m_{1k}\omega_1 + m_{2k}\omega_2)Q_{BC,2k} \quad (21.2)$

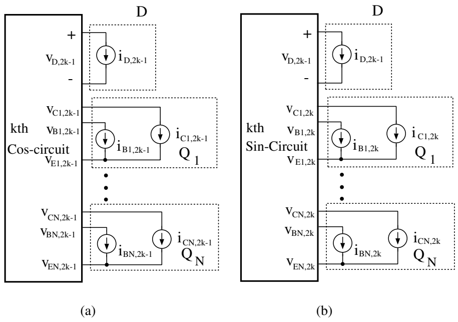

Fig.5 Intermodulation circuit of Fig. 4 at $m_{1k}\omega_1 + m_{2k}\omega_2$.

Remark that "$I's$" in (21) show the currents evaluated from (16) and "Qs" from (17) using two-dimensional Fourier transformation.

In the same way, we can calculate the above input-output two-dimensional Fourier transformations to all the nonlinear devices such diodes and MOSFETs. We also transform all the reactance elements into the linear equivalent Cosine and Sine elements [11]. Thus, the circuit Fig.4 is transformed the equivalent Cosine-Sine circuit as shown in Fig.5 at each frequency component $m_{1k}\omega_1+m_{2k}\omega_2$. It corresponds to the *determining equations of the harmonic balance method* . Thus, we can solve the circuit with DC analysis of Spice, efficiently.

4. An illustrative example

Let us calculate the frequency response curves of a mixer circuit [13] shown in Fig. 6 which has two independent frequencies $\omega_1 = 0.18[MHz]$ and $\omega_2 = 0.2[MHz]$. Although the circuit may happen to arise many frequency components, we consider DC, ω_1 ω_2, $2\omega_1$, $2\omega_2$, $\omega_1 \pm \omega_2$ components. We also set $N = K = 3$ in (2). The transistor is modeled by a Gummel-Poon model, whose parameters are given by [6]

$I_S = 10^{-14}[A]$	$\tau = 10^{-10}[sec]$	$V_t = 0.026[V]$	$\phi_B = 0.75$
$B_F = 470$	$C_{j0} = 0.1[pF]$	$V_{AF} = 150[V]$	$F_c = 0.74$
$B_R = 1$			

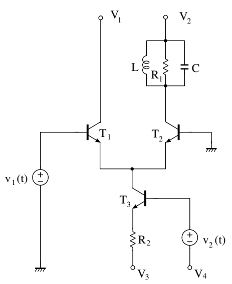

Fig.6 A differential mixer.

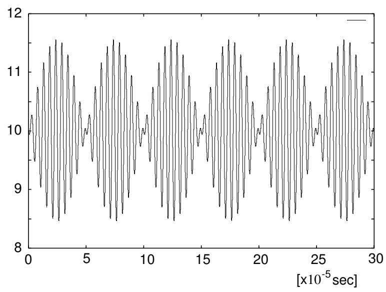

Fig.7 Transient response of the differential mixer.

$V_1 = V_2 = 10[V]$, $V_3 = -10[V]$, $V_4 = 5[V]$, $L = 10[\mu H]$, $C = 1[pF]$ $R_1 = 100[\Omega]$, $R_2 = 10[\Omega]$
$v_1(t) = 0.01 \cos 0.18 \times 10^6 t$, $v_2(t) = 0.2 \cos 0.2 \times 10^6 t$

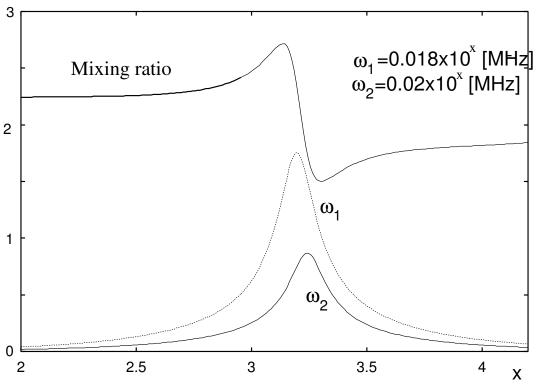

Fig.8 Frequency response curves and mixing ratio of the differential mixer.

In the numerical computation in Fig.7, we set two frequencies as follows;

$$\omega_1 = 0.018 \times 10^6 \times 10^x, \quad \omega_2 = 0.02 \times 10^6 \times 10^x$$

Thus, we have $\omega_1 = 0.18[MHz]$ and $\omega_2 = 0.2[MGz]$ at $x = 1$. Note that we take 137 [sec] to get the netlist and 463[sec] for the computation with Spice. The mixing ratio is defined by

$$M = A_1/A_2$$

for the two output amplitudes A_1 and A_2.

5. Conclusions and remarks

The distortion analysis is very important for designing the high frequency communication systems.

In this paper, we have proposed an efficient technique for calculating the frequency response curves and mixing ratio of nonlinear electronic circuits. At first, the nonlinear devices such as bipolar transistors and MOSFETs are transformed into the corresponding modules which carry out the two-dimensional Fourier transformations. Using these device modules, the nonlinear circuit is transformed into the Fourier circuit corresponding to the determining equations of the harmonic balance method, and they are solved with the DC analysis of Spice.

We have found that it will take a lot of computational time for large scale circuits containing many transistors when we apply two-dimensional Fourier transformation to them. Therefore, we need to improve the algorithm in future.

References

[1] M.Schetzen, *The Volterra and Wiener Theorems of Nonlinear Systems,* John Wiley and Sons, 1978.

[2] D.D.Weiner and J.E.Spina, *Sinusoidal Analysis and Modeling of Weakly Nonlinear Circuits with Application to Nonlinear Interference Effects,* Van Nostrand Reinhold, 1980.

[3] P.Wambacq and W.Sansen, *Distortion Analysis of Analog Integrated Circuits,* Kluwer Academic Pub., 1998.

[4] B.J.Leon and D.J.Schaefer, "Volterra series and Picard iteration for nonlinear circuits and systems," *IEEE Trans. on Circuits and Systems,* vol.CAS-25, no.9, pp.789-793, 1078.

[5] L.O.Chua and C-Y.Ng, "Frequency-domain analysis of nonlinear systems: formulation of transfer functions," *IEE Trans. Electronic Circuits and Systems,* Vol.3, no.6, pp.257-269, 1979.

[6] R.L.Geiger, P.E.Allen and N.R.Strader, *VLSI: Design Techniques for Analog and Digital Circuits,* McGraw-Hill, 1990.

[7] M.Miura, T.Shiino and K.Mori, *Circuit-Simulation Technique and MOSFET modeling*, Riaraizu Rikou Senta (Japanese), 2003

[8] T.J.Aprille and T.N.Trick, "Steady-state analysis of nonlinear circuits with periodic input," *Proc. IEEE,* vol.60, pp.108-114, 1972.

[9] K.S.Kundert, J.K.White and A.Sangiovanni-Vincentelli, *Steady-State Methods for Simulating Analog and Microwave Circuits,* Kluwer Academic Pub., 1990.

[10] A.Ushida, T.Adachi and L.O.Chua, "Steady-state analysis of Nonlinear Circuits based on hybrid methods," *IEEE Trans. on Circuits and Systems-I: Fundamental Theory and Applications,* vol.39, no.9, pp.649-661, 1992.

[11] A.Ushida, Y.Yamagami and Y.Nishio, "Frequency responses of nonlinear networks using curve tracing algorithm," *ISCAS 2002,* vol.I, pp.641-644, 2002.

[12] MicroSim: PSpice A/D Circuit Analysis User's Guides, MicroSim Co. 1995.

[13] K.K.Clarke and D.T.Hess, *Communication Circuits: Analysis and Design,* Addison-Wesley Pub. Co., 1971.

[14] A.Ushida, L.O.chua and T.Sugawara, "A substitution algorithm for solving nonlinear circuits with multi-frequency components," *Int. Jour. of Circuit Theory and Applications,*, vol.15, pp.327-355, 1987.