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Nonlinear Analysis of Charge Pump Phase-Locked Loops

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Abstract– This paper presents an analysis of the nonlinear dynamics of a Charge Pump Phase-Locked Loop. The closed-loop circuit is modelled using decision-based equations developed by Van Paemel. State space plots; bifurcation diagrams and the presence of a chaotic attractor all indicate a significant level of underlying nonlinear phenomena. A calculation of the fractal dimension of the chaotic attractor is performed, and its basin of attraction is found through simulations. Finally, the results obtained are compared to the results of a previous analysis of a different CP-PLL model.

1. Introduction

Phase-lock loops (PLLs) are closed-loop systems with negative feedback. The circuit essentially locks onto the frequency of an incoming signal and maintains this lock by extracting the phase error and aiming to reduce it to zero. This circuit is predominantly used in communication applications such as frequency synthesis and clock recovery. Recent research has seen significant attempts to better understand the complicated dynamics of PLLs through the application of nonlinear theory [1], [2].

The aim of our research is to apply nonlinear dynamical techniques to currently implemented PLLs. In most frequency synthesis applications, a Charge Pump Phase-Locked Loop (CP-PLL) is employed. Modelling of this circuit is complicated by the charge pump action, which essentially makes the circuit time varying. Two main techniques have been used to model the CP-PLL The first by Gardner [3] essentially averages the phase error output to give two equations to model the system. The second technique includes the charge-pump action, resulting in a set of decision-based equations [4].

We have previously examined the nonlinear dynamics of the Gardner CP-PLL model in [5]. This paper looks at the second approach to modelling the CP-PLL and the application of nonlinear dynamic techniques to this model. The main analysis techniques include bifurcation diagrams, state space plots and examination of basins of attraction. Finally, we will compare our findings with the results presented in [5].

2. CP-PLL Circuits and Modelling

2.1. Basic Circuit Layout

The CP-PLL consists of four major blocks, Fig. 1.

- The phase-frequency detector, which outputs a pulse proportional to the detected phase error (pulse width, τ).
- The charge-pump circuit, converting the digital signals U and D into a current, which can have three discrete values: I_p, $-I_p$ and zero
- The loop filter, converting the charge-pump current into the analog voltage V_{con}. (When combined with the charge-pump circuit provides integrating zero thus leading to zero static phase error, theoretically)
- The Voltage-Controlled Oscillator (VCO), generating an oscillating signal with a frequency controlled by the voltage V_{con}.

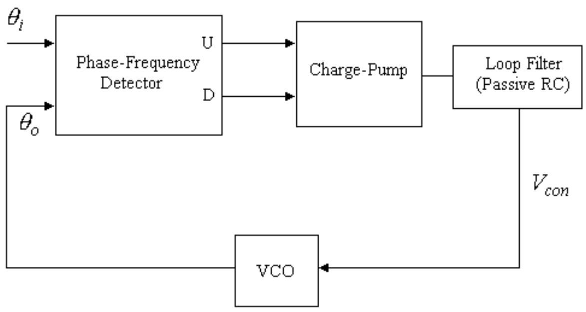

Figure 1. Block diagram of the CP-PLL circuit.

The circuit aims to match the phase of the incoming signal, θ_i, to the output from the VCO, θ_o. All four of the main circuit blocks have non-idealities, which contribute to highly nonlinear behaviour such as frequency spurs, phase jitter and cycle slipping. This has lead to increasing interest in the underlying dynamics of the CP-PLL. Basic models of PLLs have already exhibited well-documented nonlinear dynamics such as period-doubling bifurcations, limit cycles and restricted basins of attraction [1], [5].

2.2. Model of the CP-PLL Circuit

The details of this model are presented in [4]; essentially it is a decision based set of difference equations. The two state variables which fully describe the loop operation are $v(k)$ and $\tau(k)$:

***v(k)*:** This is the voltage stored on the loop filter capacitor C, after the opening of the charge pump switches.

*τ(k)***:** This is the phase detector output pulse width.

The model is flexible enough to incorporate non-idealities such as charge pump switching delays, input and output resistances, nonlinear VCO characteristic, VCO overload, frequency dependent phase detection etc. To characterize the CP-PLL, four parameters are introduced:

- I_p: the charge pump current
- R and C: the loop filter components
- K_v: the VCO gain, assumed constant

Every period T of the input signal, of frequency f_i, both state variables are calculated. In this way the state variables are updated after a fixed time interval T, resulting in a fixed sampling rate equal to the input frequency. The loop operation is therefore dependent upon the input frequency and the four previously-mentioned parameters. The difference equation for $v(k)$ is:

$$v(k+1) = v(k) + \frac{Ip}{C}\tau(k+1) \qquad (1)$$

The difference equation to calculate $\tau(k+1)$ is determined by requiring the time interval $\{T + \tau(k+1) - \tau(k)\}$ to correspond to one period of the VCO output signal, that is

$$\int_0^{T+\tau(k+1)-\tau(k)} V_{con}(t)dt = \frac{1}{K_v} \qquad (2)$$

The solution of this integral can have four different cases depending on the signs of $\tau(k)$ and $\tau(k+1)$. In addition to these four cases, two more are added to account for the out-of-lock loop behaviour. This results in an algorithm, with decision statements based on the sign and magnitude of $\tau(k)$ and $\tau(k+1)$.

Temporarily discarding the discrete-time nature of the loop, the PLL can be approximated as a continuous-time linear feedback system. The second-order PLL can then be characterized by its natural frequency F_n and its damping factor ζ [4]. These two parameters are defined as follows: the natural frequency F_n is given by

$$F_n = \frac{T}{2\pi}\sqrt{\frac{K_v I_p}{C}} \qquad (3)$$

and the damping factor ζ is defined as

$$\zeta = \frac{R}{2}\sqrt{K_v I_p C} \qquad (4)$$

3. Nonlinear Analysis of the CP-PLL Model

3.1. Fixed-Point and Stability of the Model

The fixed point of the model corresponds to $\tau(k) = 0$ and $v(k) = 1$, where $v(k)$ is normalised with respect to its steady-state value and $\tau(k)$ is normalised to the period T of the input signal. Due to the decision-based progression of the model it is not possible to analytically determine the point at which the system becomes unstable. An investigation of this model was carried out in [6], which concluded that the sequential nature of the model meant that determination of stability was only possible if the sequence were known. Using a continuous-time approximation, stability margins are however presented in [4] and are found to correspond to the results from the Gardner paper [3].

3.2. Bifurcation Diagram

For the purposes of analysis, the two continuous-time approximated parameters (F_n, ζ), defined in (3) and (4), are used as opposed to using the five circuit parameters directly. Stability plots from [4] show that the system remains stable over a range of ζ but goes unstable as F_n is increased. The bifurcation diagram, Fig. 2, clearly shows that the system remains at its fixed point for $F_n < 0.32$, and as the natural frequency is increased further period doubling bifurcations occur. This can be more clearly seen in the close-up of the bifurcation diagram, Fig. 3. After the period-doubling cascade, the system moves to another form of attractor, which is found to be chaotic in nature. This will be more closely examined in the next section.

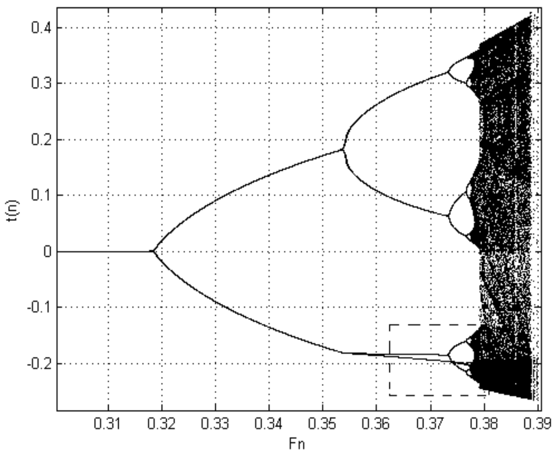

Figure 2. Bifurcation diagram, CP-PLL model, $\zeta = 0.4$.

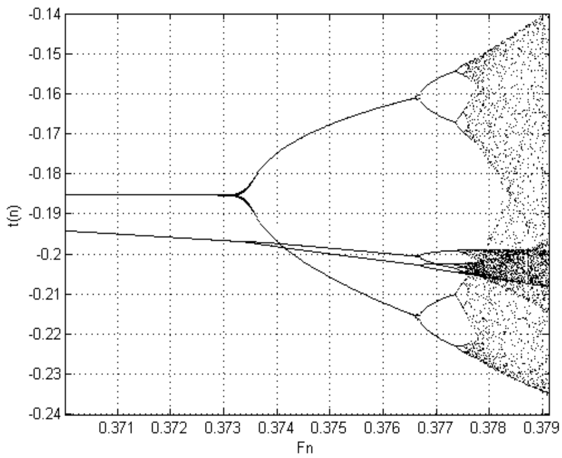

Figure 3. Close-up of the dashed region of Fig. 2

3.3. Chaotic Attractor

For the CP-PLL model, a chaotic attractor, Fig. 4, is found to exist over a range of values of natural frequency F_n and damping factor ζ. The existence of a chaotic attractor in PLL circuits has been previously observed and analysed [2]. The apparent fractal nature of the attractor can by quantified by a calculation of the correlation dimension [7] which was found to be 1.0146. Evaluation of the Lyapunov exponents confirms the attractor to be chaotic, with one being positive, the other negative.

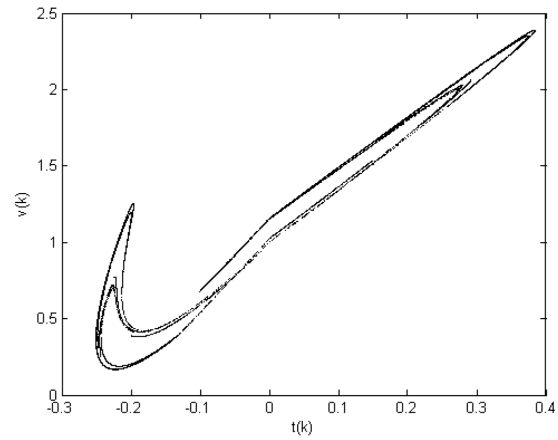

Figure 4. Chaotic attractor (F_n : 0.38 ζ : 0.4).

3.4. Basins of Attraction

The method of nonlinear dynamics used to identify the global behaviour of trajectories is the concept of basins of attraction. The basin of attraction for a particular attractor consists of the set of initial points each of which give rise to a trajectory that approaches the attractor. For the fixed point of the system, simulations indicate that it is globally attracting. The basin of attraction for the fixed point therefore corresponds to the entire state space.

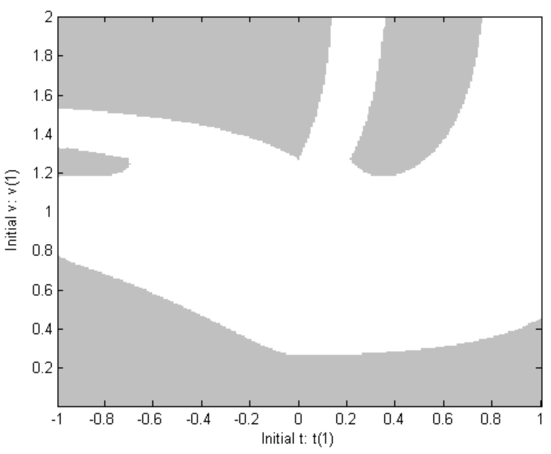

Figure 5. Basin of attraction of period-2 behaviour (white) F_n : 0.33 ζ : 0.4 The grey region corresponds to overload

However, for period-2 behaviour, the basin of attraction is found to be restricted, Fig. 5. It is found that trajectories not attracted to period-2 correspond to loop overload. This situation occurs when the output pulse from the phase detector, $\tau(k)$, is too large for the VCO to make the necessary adjustments. The model, [4], does allow for the occurrence of overload, however there are obviously limits to the rate at which the loop can adjust. For the chaotic attractor the basin of attraction, Fig. 6, was also obtained through simulations. Again, trajectories that do not go to the chaotic attractor are found to correspond to loop overload.

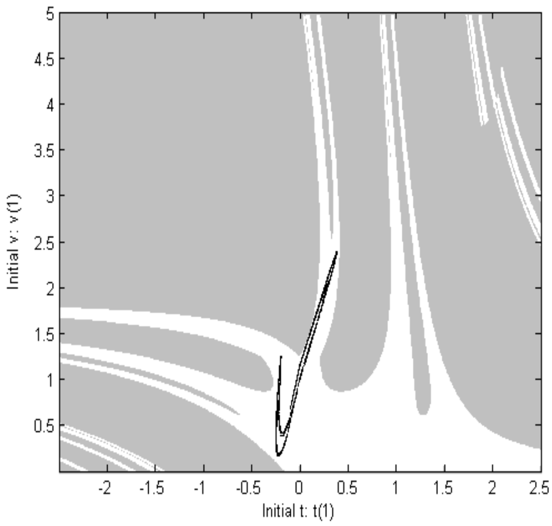

Figure 6. Basin of attraction (in white) chaotic attractor F_n : 0.38 ζ : 0.4 The grey region corresponds to overload

4. Comparison of Results for different CP-PLL Models

Having previously examined the nonlinear dynamics of another model of the CP-PLL [3] in [5], it is possible to compare and contrast these findings with those of the model [4] analysed in this paper. A stability analysis of both models gives the same result for the region of stability, [4], which is

$$C > \frac{I_p K_v T^2}{4 - 2 I_p K_v R T} \quad (5)$$

As previously discussed, the essential difference between the two models is the representation of the charge-pump action. The Gardner model [3] time-averages this effect resulting in two difference equations. An analysis of the nonlinear dynamics of this model was carried out in [5]. It was found that the fixed point of this model had a restricted basin of attraction, Fig. 7, which is fractal in nature. Examination of the equations allowed for the onset of period-2 behaviour to be determined. However, the period-doubling bifurcation is followed by a Hopf bifurcation, Fig. 8. This in turn, resulted in the emergence of a disjoint attractor.

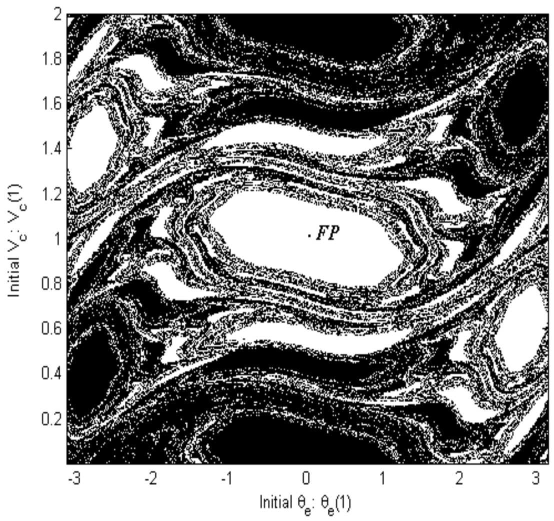

Figure 7. Basin of attraction (in white) of the fixed point for the Gardner CP-PLL model [3].

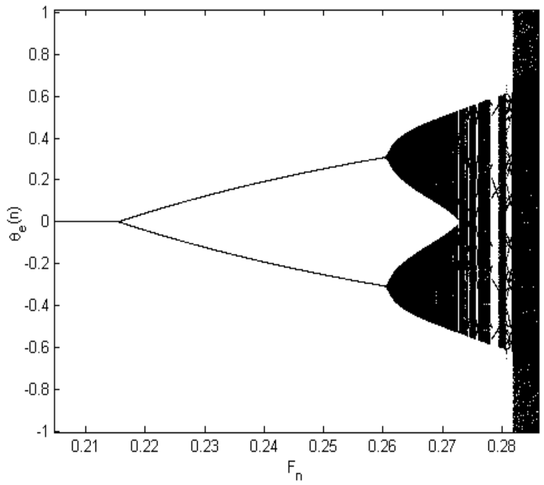

Figure 8. Bifurcation diagram of the Gardner CP-PLL model [3].

The model of the CP-PLL examined in this paper, presented by Van Paemel in [4], adopted a much different technique in its treatment of the charge-pump action. This model incorporates the time-varying nature of the charge-pump circuit, resulting in a set of nonlinear difference equations, employing an algorithmic routine to determine the next step in the iteration. The bifurcation diagram for this model, Fig. 2, clearly shows the existence of a period-doubling cascade. The fixed point is found by simulations to be globally attracting, with a restricted basin existing for period-2 and higher order behaviour. For this model, a chaotic attractor was also found to exist for a range of parameter values.

The nonlinear dynamics for the two models of the CP-PLL vary greatly. The models differ in type of bifurcation, presence of attractors and in the forms of the basins of attraction. In can be concluded for the CP-PLL, that the treatment of the charge-pump action in the model, has a large effect on the observed nonlinear dynamics of the model.

5. Conclusion

The nonlinear dynamics of a model of the CP-PLL were investigated. Bifurcation diagrams revealed a period-doubling cascade, followed by the presence of a chaotic attractor. The chaotic attractor in state space was found to be fractal in nature. A value for the correlation dimension, of the chaotic attractor was calculated and its basin of attraction was found by simulations. Finally, a comparison was made between the results obtained for this model, and results previously presented, using a different model of the CP-PLL. It was found that the representation of the charge-pump action in the model greatly affected the observed nonlinear dynamics.

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