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Chaotic Bursts in a Modified Discrete-Time Neuron Model

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Abstract—In this paper, we propose a modified bursting neuron model, which is a natural extension of an original one proposed by the author *et al.* We will show that chaotic bursts appear in the modified model though there exhibit quasi-periodic bursts in the original one. Moreover, we will show that such chaotic bursts appear by breaking up a pair of invariant closed curves, which is generated by a Hopf bifurcation for a pair of two-periodic points.

1. Introduction

When neuron activity alternates between a quiescent state and repetitive spiking(spiking state), the neuron activity is said to be a *bursting* or a *burst*. The bursting oscillations in biological neurons play an important role[1, 2].

On the other hand, many bursting neuron models have been proposed. Moreover, Izhikevich clarified a mechanism of generating the bursting in continuous-time models from bifurcation theoretical point of view[3]. In discrete-time models, Aihara *et al.* proposed chaotic neurons and neural networks[4]. Kitajima *et al.* reported that bursts are generated in the chaotic neural network with a ring structure[5] though the bursting cannot be observed in the single model because the chaotic neuron has only one-dimensional dynamics. Therefore, we proposed a high-dimensional discrete-time neuron model by extending the chaotic one[6]. In a two-dimensional model, the bursting is generated when a Hopf bifurcation for a pair of two-periodic points occurs. We also proposed a design method for the bursting neuron model with given specifications[7].

However, a neural network composed by the bursting neurons has not been studied yet. Therefore, we propose a modified bursting neuron model in this paper. The modification is introduced in order to add another internal state term which is feedback inputs from the constituent neurons in the neural network. In this paper, we consider only one modified bursting neuron model because it is important to know its dynamics primarily.

This paper is organized as follows: In Sec. 2, we introduce chaotic neurons and neural networks, and describe bursting neuron models, then, we propose a modified bursting neuron model. We also review a mechanism of bursts in the original neuron model in Sec. 2. In Sec. 3, we show that chaotic bursts appear in the proposed model, then, bifurcation analysis is applied, and we show that they are generated by breaking up a pair of two invariant closed curves which is caused by a Hopf bifurcation for a pair of two-periodic points. Section 4 concludes this paper.

2. A Modified Bursting Neuron Model

2.1. Chaotic neurons and neural networks

Aihara *et al.* proposed the following chaotic neuron[4]:

$$y(t+1) = ky(t) - \alpha f(y(t)) + c, \tag{1}$$

where t is a discrete time, $y(t)$ is an internal state at time t, α and c are parameters, and k is a damping factor of the refractoriness. The output $x(t)$ is given by

$$x(t) = f(y(t)) = \frac{1}{1+\exp(-y(t)/\varepsilon)}, \tag{2}$$

where f is the logistic function with the steepness parameter ε.

Aihara *et al.* also proposed a chaotic neural network by coupling N chaotic neurons[4]. In this paper, we assume that there are no external inputs for simplicity. Then, the dynamics of the ith chaotic neuron integrated in the chaotic neural network is described by

$$\begin{cases} y_i(t+1) = k_r y_i(t) - \alpha x_i(t) + c, \\ z_i(t+1) = k_f z_i(t) + \displaystyle\sum_{j=1}^{N} w_{ij} x_j(t), \end{cases} \tag{3}$$

where $y_i(t)$ and $z_i(t)$ are internal state terms for refractoriness and feedback inputs from the constituent neurons in the networks, respectively, k_r and k_f are the decay parameters for the refractoriness and the feedback inputs, respectively, and w_{ij} is a synaptic weight to the ith constituent neuron from the jth constituent neuron. The output $x_i(t)$ is given by

$$x_i(t) = f(y_i(t) + z_i(t)). \tag{4}$$

If the number of neurons in the chaotic neural network is 1, the network is simply described by

$$\begin{cases} y(t+1) = k_r y(t) - \alpha x(t) + c, \\ z(t+1) = k_f z(t) + w x(t). \end{cases} \tag{5}$$

2.2. Bursting neuron models

We proposed the following bursting neuron model[6]:

$$\begin{cases} y_1(t+1) = k_1 y_1(t) + k_2 y_2(t) - \alpha f(y_1(t)) + c, \\ y_2(t+1) = y_1(t), \end{cases} \quad (6)$$

where $y_1(t)$ and $y_2(t)$ are internal states, k_1, k_2, α, and c are parameters. The output $x(t)$ is given by

$$x(t) = f(y_1(t)). \quad (7)$$

If $k_2 = 0$, Eq. (6) is reduced to Eq. (1). Therefore, the bursting neuron model is one of extensions of Aihara's.

Bursting oscillations in the model can be observed by the following mechanism: There are some pairs of two-periodic points in Eq. (6). When a Hopf bifurcation for one of the pairs occurs, there appears a pair of non-isolated invariant closed curves surrounding each two-periodic point. On the bifurcation parameter, an internal state $(y_1(t),\ y_2(t))$ visits each non-isolated invariant closed curve alternately. Then, bursting oscillations can be observed through the logistic output function (2). Moreover, a necessary condition for the Hopf bifurcation is $k_2 = \pm 1$, which can be derived theoretically.

2.3. Modified bursting neuron models

If we construct a neural network with the N bursting neurons, the dynamics of the ith bursting neuron in the neural network is described by

$$\begin{cases} y_{i1}(t+1) = k_1 y_{i1}(t) + k_2 y_{i2}(t) - \alpha x_i(t) + c, \\ y_{i2}(t+1) = y_{i1}(t), \\ z_i(t+1) = k_f z_i(t) + \sum_{j=1}^{N} w_{ij} x_j(t), \end{cases} \quad (8)$$

where the output is given by $x_i(t) = f(y_{i1}(t) + z_i(t))$.

Assume that the number of the bursting neurons is 1, that is, $N = 1$. Then, the following model is derived:

$$\begin{cases} y_1(t+1) = k_1 y_1(t) + k_2 y_2(t) - \alpha x(t) + c, \\ y_2(t+1) = y_1(t), \\ z(t+1) = k_f z(t) + w x(t). \end{cases} \quad (9)$$

We call Eq. (9) *a modified bursting neuron model* in this paper. When a neural network with the bursting neurons is studied, it is important to know dynamics of the single modified neuron model. In next section, we study the modified model from bifurcational points of view.

3. Analysis

3.1. Chaotic bursts

Figure 1 shows a simulation result in the modified model with the following parameters:

$$\begin{cases} k_1 = 0.25,\ k_2 = 0.95,\ \alpha = 1,\ c = 0.5, \\ \varepsilon = 0.04,\ k_f = 0.3,\ w = 0.3. \end{cases} \quad (10)$$

Since $k_2 \neq \pm 1$, no bursting oscillations occur in the original model (6). In the modified model, however, Fig. 1 obviously shows that a bursting oscillation appears.

Figures 2 and 3 show a trajectory of the burst in the phase space and its projection to a y_1-y_2 plane, respectively. Figure 4 is enlargement of the square region in Fig. 3, and Fig. 5 is also enlargement of Fig. 4. Figure 5 shows that there is a strange attractor.

Therefore, this bursting oscillation is *chaos*. To best of our knowledge, such a *chaotic* burst cannot be observed in the original bursting neuron model (6).

3.2. Bifurcation analysis

As we reviewed a mechanism of generating bursting oscillations in the original model in Sec. 2.2. Since a pair of invariant closed curves, which is generated by a Hopf bifurcation for a pair of two-periodic points, plays an important role for generating bursting oscillations, a pair of two-periodic points in the modified model will be also important to generate the chaotic burst. Therefore, we apply bifurcation analysis into the modified model.

Figure 6 shows a bifurcation set of the Hopf bifurcation. There exist the stable two-periodic points in a downward area in Fig. 6 though they are unstable in the upward area.

When the parameters are set to be Eq. (10), there exists a pair of unstable two-periodic points denoted by P_1 and P_2. Each two-periodic point is surrounded by each chaotic attractor shown in Fig. 2. Their eigenvalues are given by

$$0.3244,\ 0.3800 \pm 1.2730\mathrm{j}. \quad (11)$$

Therefore, P_1 and P_2 are united by both a one-dimensional stable node and a two-dimensional unstable focus.

On the other hand, Fig. 7 shows a power spectrum of the chaotic burst in Fig. 1. Note that the analyzed data is given by every other time because the chaotic attractor is constructed by two pieces and that the number of the data is 4096. Figure 7 has a peak with about 1.2 [rad/Hz]. This angular frequency is approximately derived by eigenvalues of P_1 and P_2. Since P_1 and P_2 have a pair of complex conjugate eigenvalues, the argument of them θ is given by

$$\theta = \tan^{-1}\frac{1.2730}{0.3800} = 1.281\ \text{[rad]}. \quad (12)$$

Therefore, the chaotic burst is influenced by P_1 and P_2, which are unstabilized by the Hopf bifurcation for them.

3.3. A route to the chaotic bursts

Figure 8 shows a bifurcation diagram by increasing the parameter k_2 from 0.8 to 0.95 while the other parameters are fixed to Eq. (10). Figures 9(i)-(vi) also show attractors projected to a y_1-y_2 plane when k_2 is 0.827, 0.845, 0.890, 0.900, 0.920, and 0.950, respectively. Note that only one piece is shown in each figure because each attractor has two pieces.

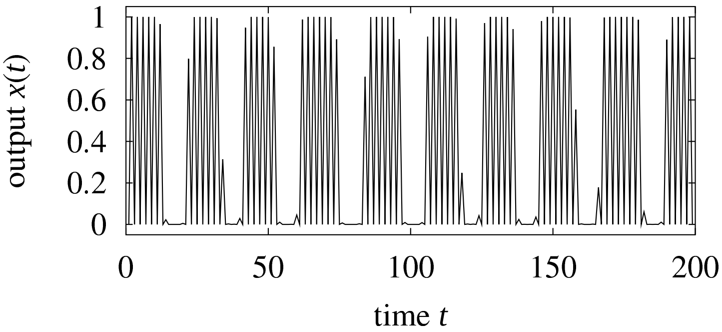

Figure 1: A bursting oscillation.

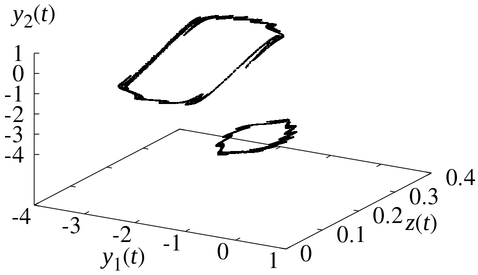

Figure 2: A trajectory.

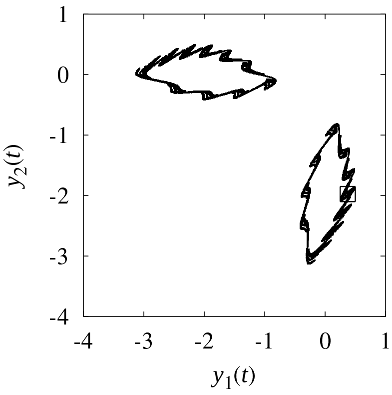

Figure 3: A projection to a y_1-y_2 plane.

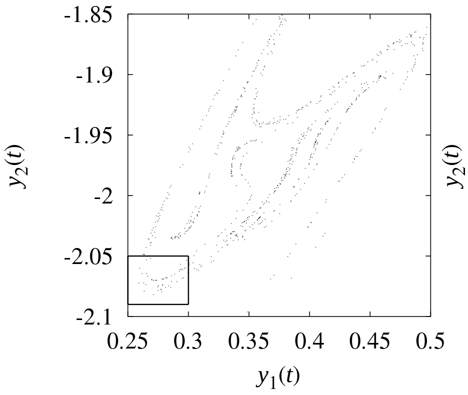

Figure 4: Enlargement of Fig. 3.

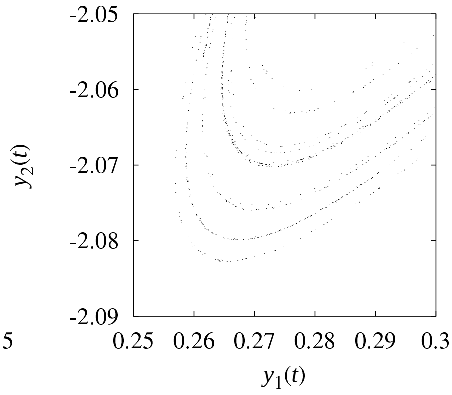

Figure 5: Enlargement of Fig. 4.

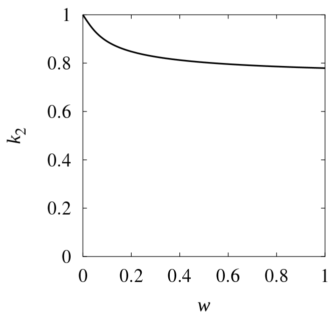

Figure 6: A bifurcation set.

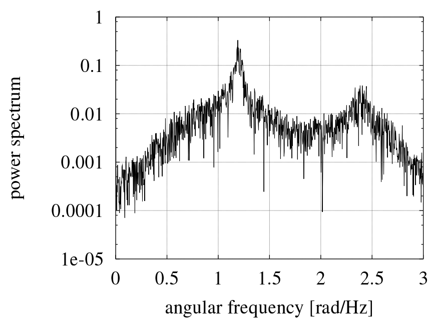

Figure 7: A power spectrum.

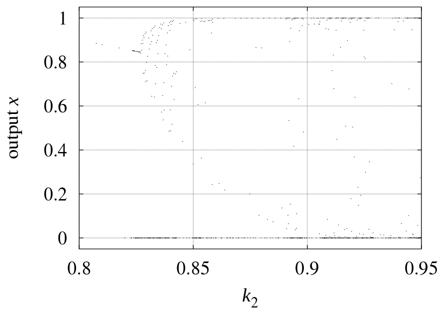

Figure 8: A bifurcation diagram.

The Hopf bifurcation for P_1 and P_2 occurs when k_2 is about 0.8262. After the bifurcation, a pair of invariant closed curves appears. However, Fig. 9(i) shows that they still remain when k_2 increases. When k_2 increases more, phase lockings sometimes occur shown in Figs. 9 (ii) and (iv), and the invariant closed curves with folds appear shown in Figs. 9 (iii), (v), and (vi). Therefore, such a chaotic burst is generated by breaking up the invariant closed curves, which result from the Hopf bifurcation for P_1 and P_2.

4. Concluding Remarks

In this paper, we proposed a modified bursting neuron model based on the original one. This model is a natural extension in order to construct a neural network by coupling the original ones.

We showed that there exhibit *chaotic* bursts in the modi-

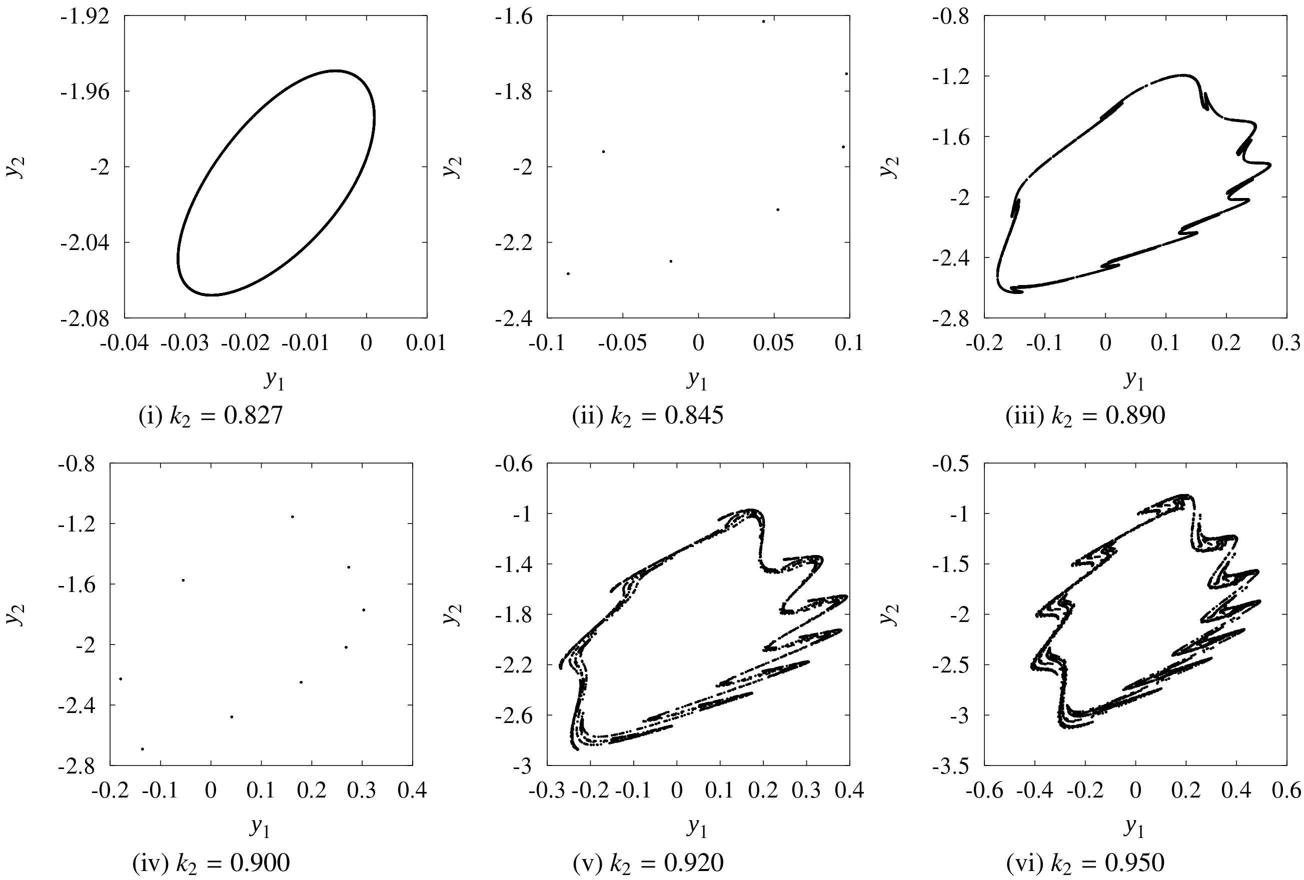

Figure 9: Attractors.

fied model though there only exist quasi-periodic bursts in the original one. We also showed that such chaotic bursts appear by breaking up a pair of invariant closed curves generated by a Hopf bifurcation for a pair of two-periodic points.

One of our future studies is analysis of a neural network with the modified bursting neurons.

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