2005 International Symposium on Nonlinear
Theory and its Applications (NOLTA2005)
Bruges, Belgium, October 18-21, 2005

Small-World Networks Produced by Optimization Problems

Hideki NAKAYAMA, Hiroto TANAKA, and Toshimitsu USHIO

Graduate School of Engineering Science, Osaka University
1–3 Machikaneyama, Toyonaka, OSAKA, 560–8531, Japan
Email: nakayama@hopf.sys.es.osaka-u.ac.jp

Abstract—In this paper, we consider a topology of complex networks. Recently, a topological property of such networks called a "small world" has been paid much attention to. Some models generating the small-world networks have been proposed. However, they are some idealized assumptions. We will propose a new model which is more realistic and show that the model will generate some kinds of the small-world properties.

1. Introduction

Recently, a topology of many complex networks has been paid much attention to. A "small world" introduced by Watts and Strogatz is one of topological properties of the networks[1]. Small-world networks are highly clustered of regular lattices and have a short average distance. On the other hand, a "scale free" is introduced to topological properties by Barabási *et al.* [2]. Scale-free networks are characterized by a node degree distribution that decays as a power low.

In order to classify graphs' topological properties, Watts and Strogatz defined two characteristic quantities[1]: One is an average distance, called *a characteristic path length* L, which shows a global property on the graph. The other is *a clustering coefficient* C, which indicates a local property. They defined the small-world networks by using C and L.

Some models generating the small-world networks have been proposed[1, 3, 4, 5]. However, they have some idealized assumptions, which are merely satisfied in the real world. In this paper, we will propose a more realistic model.

This paper is organized as follows: In Sec. 2, we review studies of small-world networks and models generating them, then we point out issues in the models. In Sec. 3, we propose a new model. In Sec. 4, we show simulation results of the proposed model. Finally, in Sec. 5, we conclude this paper.

2. Reviews of the Small-World Networks

Watts and Strogatz defined two characteristic quantities [1]: *a characteristic path length* L and *a clustering coefficient* C. L is defined by an averaged shortest path length between all pairs of nodes. C is defined as follows: Suppose that a node v has k_v neighbors. Then at most $k_v(k_v-1)/2$ edges can exist between the neighbors. Denoted by C_v is the fraction of these allowable edges that actually exist. Define C as the average of C_v over all v.

Consider a graph G with n nodes. A graph $G_{\rm rand}$ is called *a random graph* of G if it has the same number of total nodes and edges as G and it's edges are wired at random. By the use of L and C, Watts and Strogatz defined the small world as follows [3]: $n \gg k_{\max} \gg 1$, $L \approx L_{\rm rand}$, $C \gg C_{\rm rand}$, where $L_{\rm rand}$ and $C_{\rm rand}$ are a characteristic path and a clustering coefficient of a $G_{\rm rand}$, respectively, and $k_{\max}$ is a maximum number of each edge in G.

Amaral *et al.* considered small-world networks from the viewpoint of a node degree distribution and introduced three classes of small-world networks[6]: (a) scale-free networks, (b) broad-scale networks, (c) single-scale networks. A scale-free network is characterized by a node degree distribution that decays as a power law. A broad-scale network is characterized by a node degree that has a power law regime followed by a sharp cutoff. A single-scale network is characterized by a node degree distribution with a fast decaying tail. Amaral *et al.* considered that such different networks result from limitation of adding new edges. Note that small-world networks are defined by only L and C though scale-free ones are defined by shape of a node degree distribution. Therefore, small-world networks may not be scale free, and *vice versa*.

Some models generating small-world networks have been proposed. Watts and Strogatz[1, 3] considered a ring lattice, which has n nodes arranged at regular intervals on the ring, with each node connected to its k nearest neighbors. This graph is called *a k-regular graph*. With a probability p, disorder is introduced into the graph by randomly rewiring each edge. While the graph remains k-regular at $p = 0$, a random graph appears at $p = 1$. When p lies in the range $(0, 1)$, small-world graphs are generated. Graphs with a few short cuts are typical examples of the small world.

Mathias *et al.* considered physical networks, in particular neural or transportation networks[4]. They pointed out that the small world arises from the trade-off between maximization of connectivity in a graph and minimization of cost in the physical world. They considered two kinds of distance: the characteristic

path length L and a physical distance W. The smaller L is, the more efficient transmission of the signal, information, the material, and so on is. On the other hand, the smaller W is preferable because a wiring cost of edges will be proportional to the physical distance. Mathias *et al.* defined the following trade-off function E between L and W:

$$E = \lambda L + (1 - \lambda)W, (0 \leq \lambda \leq 1). \quad (1)$$

They also considered a k-regular graph with n nodes. Each edge is rewired in order to minimize E. While the graph remains k-regular at $\lambda = 0$, a random graph appears at $\lambda = 1$. When λ lies in the range $(0, 1)$, small-world graphs can be generated.

Davidsen *et al.* considered social networks, in particular acquaintanceship[5]. They reported that the small world appears in local interactions that people introduces his/her acquaintance to another acquaintance. They considered a graph with n nodes. In the graph, each node indicates a person and an edge between nodes i and j exists if the persons i and j know each other. A randomly chosen person picks up his/her two acquaintances at random and introduces them to each other. If they have not met before, a new edge between them is wired. In the case a chosen person has less than two acquaintances, he/she introduces himself/herself to another person at random. In addition, with a probability p, a randomly chosen person and his/her all own edges are removed from the network, then, a new person with one randomly chosen acquaintance is added to the network. After these steps are iterated, a small-world with a scale-free property graph can be generated.

Though such models are theoretically comprehensible, they have some idealized assumptions. First of all, it is unnatural that all nodes have been already given. Many actual networks grow up by adding nodes and become large scale. Furthermore, as models of Watts and Strogatz[1, 3] or Davidsen *et al.*[5], edges will not be randomly wired. For example, it is said that there are at least 8×10^8 documents in World Wide Web, and it is impossible to optimize the whole network like Mathias *et al.*[4]. If one wants to add a new node into the World Wide Web, one cannot obtain information of all nodes and edges. It is realistic that one wants to wire a node to nodes such that he/she gets a best profit based on only local information.

For such reasons, we propose a new model with the following properties in the next section:

- A considered graph is growing-up by adding new nodes.
- When the graph grows up, and an added node uses its local information.
- The wiring problem is formulated as an optimization one with respect to the added node.

3. A Proposed Model

In this paper, we propose a new model with the following properties: Consider a circumference of a circle and divide it into M pieces. Each divided piece is called a *block* and can contain at most one node. In this model, each node is added one by one. An added node enters a vacant block randomly. The added node is assigned to a value and a cost bound. When it is wired to another node, it pays cost based on the physical distance connecting between the nodes. The added node chooses a set of nodes to be wired in order to maximize the sum of values under the limitation of the assigned cost bound. In addition, a node which has already existed evolves with a probability p. The evolved node gets a value and a cost bound further and is connected with other nodes. We propose the following model for generating a graph:

1. **Definition**

   - M: Number into which the circumference is divided.
   - N: The number of nodes.
   - W: A set of values.
   - C: A set of cost bounds.
   - p: An evolutional number.
   - rand_1: A random number in the interval $[0,\ 1]$.

2. **Main**

   (a) A new node 1 is added to the graph and gets a value w_1 and a cost bound c_1 from W and C, respectively. The node 1 is randomly assigned to a vacant block.

   (b) $i = 2$: Node number.

   (c) while($i \leq N$)

      i. if($\text{rand}_1 < p$)

         A. A new node i is added to the graph.

         B. The node i gets a value w_i and a cost bound c_i from W and C, respectively.

         C. The node i randomly is assigned to a vacant block.

         D. Calculate each distance between itself and a node j $(1 \leq j \leq i - 1)$.

         E. The node i solves a wiring problem and gets a set of other nodes to be wired.

         F. The node i is wired to all nodes in the set and pays a cost corresponding to a sum of the distances.

G. $i := i + 1$.

ii. else

A. Pick up a node j at random ($1 \leq j \leq i-1$).

B. The node j gets a value and a cost bound further, that is, w_j and c_j are reassigned.

C. The node j solves a wiring problem and gets a set of other nodes to be wired. However only adding edges are allowable, that is, removing edges is not permitted.

D. The node j is wired to all nodes in the set and pays a cost corresponding to the sum of distances.

3. **End**

- Output information of edges.

Figure 1 shows an example. A node i's block number is defined by x_i, and the distance between nodes i and j is $d_{ij} = d_{ji} = |x_i - x_j|$. If an egde between the nodes exists, $e_{ij} = e_{ji} = 1$ and otherwise $e_{ij} = e_{ji} = 0$. A wiring problem with respect to the added node $i (i \geq 2)$ in the main part of the algorithm is equivalent to the following optimization problem:

$$\text{Maximize} \sum_{\ell=1}^{i-1} e_{i\ell} w_\ell, \tag{2}$$

$$\text{subject to} \sum_{\ell=1}^{i-1} e_{i\ell} d_{i\ell} \leq c_i. \tag{3}$$

Since an added node i cannot connect a node ℓ such that $d_{i\ell} > c_i$, the node i does not need to collect information of the node ℓ. That is, the node i use information of a node j such that $d_{i\ell} \leq c_i$. Therefore, node i use *local* information of itself.

This optimization problem is a 0-1 knapsack problem that is a famous NP one. Since it is impossible to solve an optimal solution in realistic time, we derive an approximated solution by using some strategies. We investigate the topological difference among the obtained solutions derived by each strategy.

We consider four strategies as follows:

- **Random Strategy**: An added node i picks up nodes randomly within the range of its cost bound c_i. Since this strategy is based on randomness, an answer of each optimization problem (2)(3) is assigned by the best result, which is derived in R times trial.

- **Cost Priority Strategy**: An added node i sequentially chooses from nearer nodes within the range of its cost bound c_i.

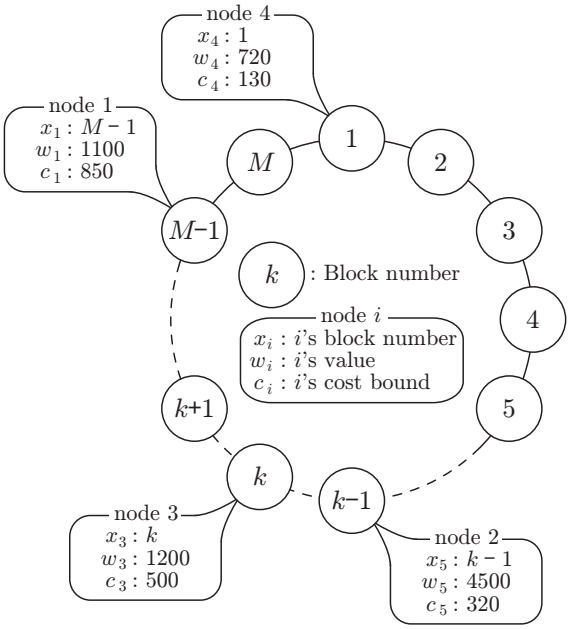


Figure 1: An example of algorithm.

- **Value Priority Strategy**: An added node i sequentially chooses from more valuable nodes within the range of its cost bound c_i.

- **Greedy Strategy**: An added node i sequentially chooses from nodes with higher value per distance within the range of its cost bound c_i.

4. Simulation Results

Let M, N, and R be 1000, 3000, and 100, respectively, W and N are uniform distributions from 1 to 10000 and 1 to 2000, respectively, and p is defined as follows:

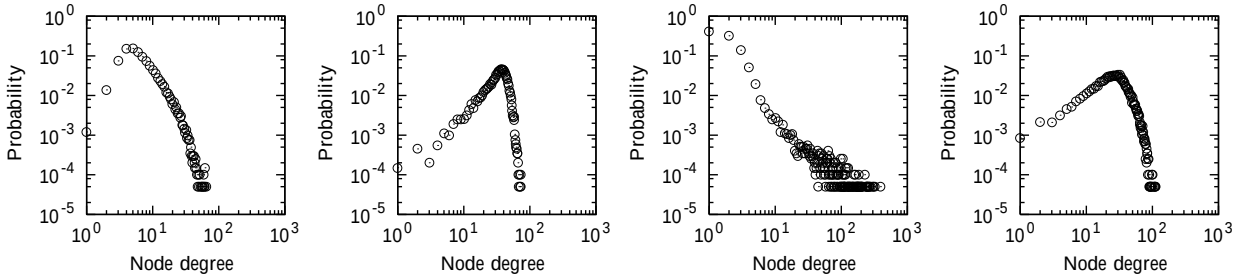
$$p = \alpha \frac{i}{N-1}, \tag{4}$$

where i is a number of nodes which have already existed and α is set to be 0.1.

Table 1 and Figs. 2 and 3 show simulation results by the four different strategies. Table 1 shows that all graphs obtained by the four strategies have small-world properties. However, obtained node degree distributions are different. In Fig. 2, all distribution plots are averaged over 100 simulations. In particular, distributions of the random strategy and the value priority one show scale-free network's properties though the other strategies imply single-scale network's ones.

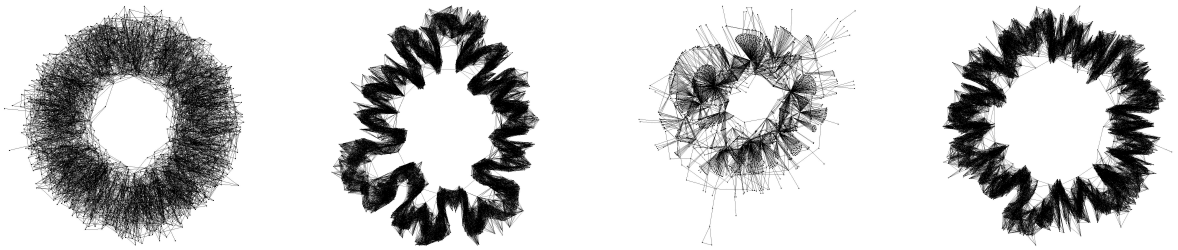
5. Concluding Remarks

We have proposed a novel growing-up network model based on local information of an added node.



(a) Random Strategy. (b) Cost Priority Strategy. (c) Value Priority Strategy. (d) Greedy Strategy.

Figure 2: Node degree distributions.



(a) Random Strategy. (b) Cost Priority Strategy. (c) Value Priority Strategy. (d) Greedy Strategy.

Figure 3: Obtained graphs.

Table 1: Simulation Results.

	L	C	L_{rand}	C_{rand}
Random strategy	5.56	0.0714	3.85	0.00384
Cost priority strategy	7.13	0.670	2.51	0.0168
Value priority strategy	4.64	0.360	6.03	0.000953
Greedy strategy	7.10	0.632	2.60	0.0149

The model partly contains 0-1 knapsack problems, which are approximately solved by four different strategies. The numerical results have shown that two of the strategies produce scale-free networks and that the others generate single-scale ones. We can get small-world networks but different strategies can generate different kinds of the small world.

Acknowledgments

This research is financially supported by a foundation Osaka University Supporters' Association.

References

[1] D. J. Watts, and S. H. Strogatz, "Collective dynamics of 'small-world networks'", *nature*, vol. 393, pp. 440-442, 1998.

[2] A. Barabási, and R. Albert, "Emergence of scaling in random networks", *Science*, Vol. 286, pp. 509-512, 1999.

[3] D. J. Watts, "Networks, Dynamics, and the Small-World Phenomenon", *Proc. Nat. Acad. Sci. U.S.A.*, vol. 105, no. 2, pp. 493-527, 1999.

[4] N. Mathias, and V. Gopal, "Small worlds: How and why", *Physical Review E*, Vol. 63, No. 2, pp. 021117-1-021117-12, 2001.

[5] J. Davidsen, H. Ebel, and S. Bornholdt, "Emergence of a Small World from Local Interactions: Modeling Acquaintance Networks", *Physical Review Letters*, Vol. 88, No. 12, pp. 128701-1-128701-4, 2002.

[6] L. A. N. Amaral, A. Scala, M. Barthélémy, and H. E. Stanley, "Classes of small-world networks", *Proceedings of the National Academy of Sciences*, Vol. 97, No. 21, pp. 11149-11152, 2000.