

Oscillator of Quasiperiodic Oscillations. Two-dimensional Torus Doubling Bifurcation

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Abstract—We propose a new autonomous differential dynamical system with dimension $N = 4$, whose solution represents stable two-frequency oscillations. It is shown that the system realizes a sequence of period doubling bifurcations of two-dimensional ergodic tori. It is established that when the doubling bifurcation takes place, no resonances on a torus are observed and the ergodic torus is doubled.

1. Introduction

The study of bifurcations of quasi-periodic oscillations and transitions to chaos via their destruction is one of interesting problems of nonlinear dynamics. There are well-known mechanisms, such as the Landau–Hopf scenario [1, 2], the Ruelle–Takens scenario [3, 4], and the Afraimovich–Shilnikov scenario [5], that describe the details of transitions to chaos through multi-frequency oscillations. This problem is analysed in many works where a wide class of real and model dynamical systems are used [6]. With this, regimes of quasi-periodic oscillations are most frequently observed in periodically driven discrete-time and differential systems.

It seems to be quite interesting to develop the simplest autonomous differential system that can generate a solution in the form of stable two-frequency oscillations and demonstrate the basic bifurcation mechanisms of their destruction including period doubling bifurcations. In spite of the fact that the two-dimensional torus doubling was discovered many years ago [7, 8, 9], the details of the bifurcation mechanism of ergodic torus doubling still remain unclear up to now.

A two-dimensional torus can be realized in a three-dimensional autonomous dissipative system as it has been shown, for example, in [10]. However, the implementation of torus doubling bifurcation requires the system dimension to increase to $N \geq 4$.

In the present paper we propose the simplest autonomous dynamical system with dimension $N = 4$ that can realize the regime of a stable two-dimensional torus and demonstrate torus doubling bifurcations as well as transitions to chaos via torus breakdown. The proposed model is used to examine certain details of the bifurcation mechanism of two-dimensional torus doubling.

2. Model of the Oscillator

As an initial system, we consider the known model of Anishchenko–Astakhov’s oscillator [11] that reads:

$$\begin{aligned} \dot{x} &= mx + y - xz - dx^3, \\ \dot{y} &= -x, \\ \dot{z} &= -gz + g\Phi(x). \end{aligned} \quad (1)$$

The first two equations of system (1) describe the Van der Pol oscillator. The third equation of system (1) represents an inertial cascade of additional feedback that includes a nonlinear convertor $\Phi(x)$ being given by a function in the form $(\exp(x) - 1)$ or $I(x)x^2$, where $I(x) = 1$ for $x > 0$ and $I(x) = 0$ for $x \leq 0$. The presence of inertial feedback serves as the major reason for appearing chaotic oscillations.

To achieve the formulated purpose, we change the inertial cascade of additional feedback that causes the dimension of the equations to increase:

$$\begin{aligned} \dot{z} &= \varphi, \\ \dot{\varphi} &= -\gamma\varphi + \gamma\Phi(x) - gz. \end{aligned} \quad (2)$$

Here, γ is the parameter of damping of the new filter, and g is the parameter characterizing its inertia. Equations (2) represent the equation of a dissipative circuit in the regime of forced oscillations:

$$\ddot{z} + \gamma\dot{z} + gz = \gamma\Phi(x). \quad (3)$$

As our investigations have shown, the regime of undamped autonomous oscillations can be obtained if the derivative $\dot{z}(t) = \varphi(t)$ is used instead of the controlling signal $z(t)$ (see Eq. (1)). In this case the equations of a new oscillator can be written down as follows:

$$\begin{aligned} \dot{x} &= mx + y - x\varphi - dx^3, \\ \dot{y} &= -x, \\ \dot{z} &= \varphi, \\ \dot{\varphi} &= -\gamma\varphi + \gamma\Phi(x) - gz. \end{aligned} \quad (4)$$

System (4) is a nonlinear dissipative dynamical system of dimension $N = 4$ and is characterized by four controlling parameters. They are m being the parameter of excitation, d being the parameter of nonlinear dissipation, γ being the parameter of damping, and g being the parameter of filter inertia.

3. Numerical simulation

It has been numerically established that system (4) can realize the regime of a stable two-dimensional torus, a torus doubling bifurcation and torus breakdown with the transition to chaos. Besides, for small values of the parameter g (when system (4) is similar to system (1)), system (4) can demonstrate period doubling bifurcations of limit cycles and the transition to chaos as in Anishchenko–Astakhov’s oscillator.

Figure 1,a shows the bifurcation diagram of system (4) modes on the plane of the controlling parameters m and g for fixed values $\gamma = 0.2$ and $d = 0.001$. The function $\Phi(x)$ is defined in the form $I(x)x^2$.

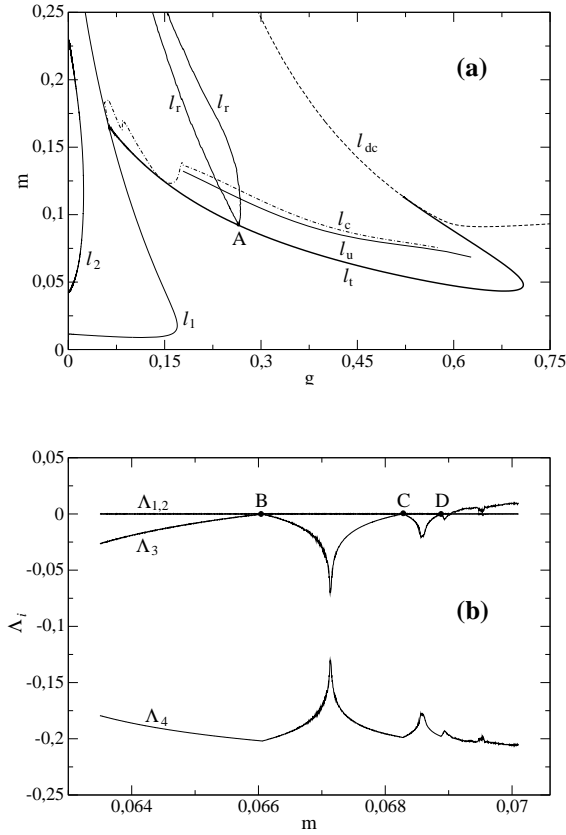


Figure 1: (a) Bifurcation diagram of regimes in system (4) for $\gamma = 0.2$ and $d = 0.001$. $l_{1,2}$ are cycle period doubling bifurcation lines, l_t is the torus birth line, l_u is the torus breakdown line, l_c is the chaotic attractor destruction line, l_r are the lines bounding the resonance $1 : 4$ region on a torus, l_{dc} are the multiple cycle lines, and A is a codimension 2 point corresponding to the condition $\phi = 1 : 4$. (b) Dependence of the Lyapunov exponent spectrum on m at $d = 0.001$, $\gamma = 0.2$, and $g = 0.5$ within the parameter range between lines l_t and l_u . B , C and D are the bifurcation points of torus period doubling

According to the Andronov–Hopf bifurcation, a stable limit cycle T_0 is born on the line $m = 0$ and undergoes the period doubling bifurcation when crossing the bifurcation line l_1 . On the curve l_2 the period doubling bifurcation is realized for the cycle that has appeared on the line l_1 (Fig. 1,a). The bifurcation line l_t corresponds to the condition when a pair of complex-conjugate multipliers of the cycle T_0 go out to the unit circle and a two-dimensional torus is softly born ($\mu_{1,2} = \exp(\pm j\phi)$). Naturally, when moving along the line l_t the angle ϕ will run a range of rational values that correspond to resonances on the torus. Figure 1,a exemplifies the resonance $\phi = 1 : 4$ region that is bounded by the lines l_r and based on the point A of codimension 2. Above the torus birth line there exists the line l_u . When crossing l_u from the bottom upwards, the transition to chaos is observed via destruction of quasi-periodic oscillations. The chaotic attractor that has appeared on the line l_u undergoes a crisis on the line l_c . The line l_{dc} corresponds to the bifurcation of merging and of consequent disappearance of a pair of saddle cycles. The bifurcation of two-dimensional torus period doubling can be observed in the region located between the lines l_t and l_u .

Now we fix $g = 0.5$, $d = 0.001$ and $\gamma = 0.2$ and consider the evolution of the torus regime in the parameter m range between the indicated lines l_t and l_u . Figure 2 demonstrates projections of attractors on the plane at the bifurcation points of the two-dimensional torus period doubling. The torus period doubling can be clearly recognized by the structure of the Poincare section as well as by analysing time series and their power spectra. In this case the torus period doubling bifurcation is defined as a modulation period doubling bifurcation (or a cycle period doubling bifurcation in the Poincare section).

From a viewpoint of the theory of bifurcations, it is quite important to answer the following question. Is an ergodic torus doubled or near the bifurcation point, first we observe a resonance on a torus and then period doubling of the resonant cycle from that a doubled torus is arised? To get the answer, we calculate a full spectrum of Lyapunov exponents at the bifurcation points, that is shown in Fig. 1,b.

As can be seen from Fig. 1,b, at the bifurcation points B , C and D three largest Lyapunov exponents become zero ($\Lambda_1 = \Lambda_2 = \Lambda_3 = 0$). The bifurcational transition is characterized by the following change of the Lyapunov exponent spectrum signature:

$$0, 0, -, - \implies 0, 0, 0, - \implies 0, 0, -, -$$

torus₂ torus₃ torus₂

The calculations were performed with a very small step on the parameter m ($\Delta m = 3 \times 10^{-6}$) and testify that no limit cycle birth is observed when passing through the bifurcation point (the Lyapunov exponent spectrum is $0, -, -, -$). At the bifurcation point we deal with a structurally unstable three-dimensional torus that gives rise to a stable doubled ergodic two-dimensional torus. Thus, the presented findings confirm once again the results described in [12] where

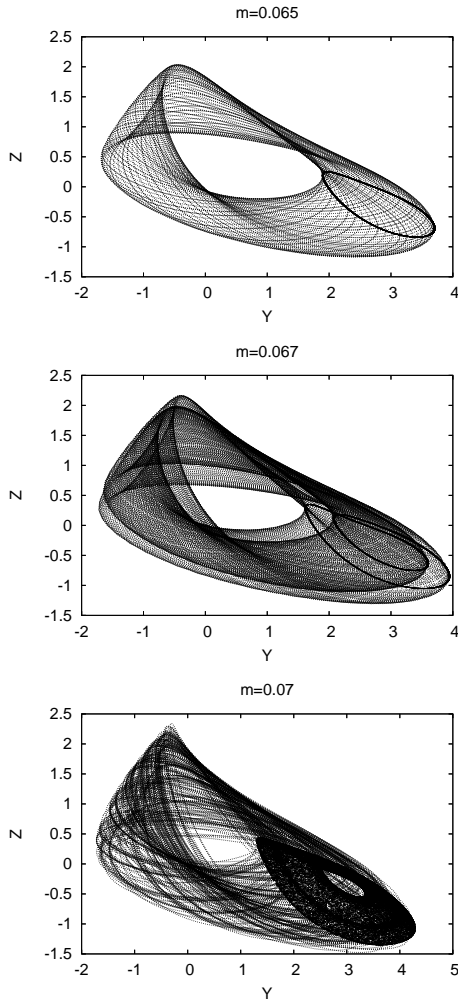


Figure 2: Projections of attractors in system (4) and of the corresponding Poincare sections on the plane for different values of the parameter m . The other parameters are $d = 0.001$, $\gamma = 0.2$ and $g = 0.5$

this conclusion was first made. The ergodic torus undergoes the period doubling bifurcation and no resonant cycles have been detected in numeric simulation.

Hence the new dynamical system (4) introduced in this paper really includes the possibility of realizing a stable regime of ergodic two-frequency oscillations (the two-dimensional torus mode) and demonstrates modulation period doubling bifurcations (two-dimensional torus period doubling). The proposed autonomous system (4) is the simplest one to explore bifurcations of quasi-periodic oscillations with two independent frequencies. The detailed analysis of the observed phenomena was not an objective of the current work but will be done in near future.

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