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Scale-free networks on a geographical planar space for efficient ad hoc communication

YukioHayashi† and *JunMatsukubo*‡

†School of Knowledge Science, Japan Advanced Institute of Science and Technology
1-1 Asahidai, Nomi-city, Ishikawa 923-1292, Japan
‡Department of Electronics and Control Engineering, Kitakyusyu National College of Technology
5-20-1 Shii, Kokuraminami, Kitakyushu-city, Fukuoka 802-0985, Japan
Email: yhayashi@jaist.ac.jp, jmatsu@kct.ac.jp

Abstract—On the commonly found scale-free (SF) structure in many complex systems, we propose a geographical configuration of SF networks without crossing links. We show the comparative short links and small number of hops, and find new relations of the average distance $\langle D\rangle \sim \log N^{\beta_d}$ and the number of hops $\langle L\rangle \sim N^{\alpha_l}$ for the network size N. These properties in the evolutional networks based on local rules are useful for efficient communication.

1. Introduction

Complex networks have been studied with great interest inspired from physics to biology, computer science, and other fields, since the **surprisingly common topological structure** called *scale-free* (SF) have been found in many real systems [2]. The degree distribution follows a power law, $P(k) \sim k^{-\gamma}$, $2 < \gamma < 3$; the heterogeneous network consists of many nodes with small degree and a few hubs with large degree, and has **good properties for efficient communication (short path length) and robust connectivity** [1]. Moreover, the restriction of links has been observed, e.g. Internet at both router and AS levels [14], road networks, and flight-connection in a major airline [6]. Indeed, the distribution of link lengths was inversely proportional to the lengths [14], excluding the Waxman's exponentially decay rule which is widely used in traffic simulations. In this paper, we consider **geographical SF network models** for a number of research fields including urban planning, electric circuits, distributed robots, sensor networks, communication networks [7], and so on.

In the state-of-the-art, a few geographical SF network models have been studied with theoretical analysis in the evolutional mechanisms of power law behavior. They are the modulated Barabási and Albert (BA) model [10][13], SF networks embedded on lattices [3], and Apollonian networks[5][15]. The 1st model is grown by introducing a new node at each time whose position is random on the Euclidean space, and the probability of connection is proportional to $k_i l^{\alpha}$, where l is the Euclidean distance between the tth (birth at time t) and the ith node with degree k_i, and α is a parameter. The case of $\alpha = 0$ is the original BA model [2]. The 2nd model is constructed on a lattice by adding the connections between the node j with randomly assigned degree k_j according to a given power law distribution $P(k)$ and the neighbor nodes within the radius $r(k_j)$ proportional to the degree k_j. In the 1st and 2nd models, crossing links cause **a serious problem such as interference of the wireless beam**, while the 3rd model is planar without crossing links; the space-filling packing is based on triangulation by adding a new node at a random position. Planar triangulation is a reasonable mathematical abstraction of ad hoc networks [7], in addition, a memoryless, no defeat, and competitive online routing algorithm has been developed [4] for such networks[1]. On the other hand, the random Apollonian networks (RAN) have long-range links which cause **dissipation of the beam power or the construction cost of links**. There exists a trade-off. Thus, we consider a new model to avoid long-range links preserving the SF structure on a planar space.

2. Delaunay-like SF networks

The Delaunay triangulation, which is the dual of a Voronoi diagram, has good properties and is useful in practical applications for geographical information processing and computer graphics [12]. In the 2-dimensional case, it is the optimal triangulation with respect to the maximin angle criterion, the minimax circumscribed circle criterion and some other criteria [8]. For example, the shortest path length between any two nodes on a Delaunay graph is **of the same order as the direct Euclidean distance**, since the ratio of them is bounded with a constant [9]. One of the fundamental techniques for equipping such properties is diagonal flipping. By using this technique to reduce long-range links, we propose a modified model from RAN with a power law degree distribution. The main idea is based on a strategy of the connecting nodes in distances as short as possible. The network is grown as follows.

[1] This sentence means that the algorithm can find a path (no defeat) using only local information (memoryless) about the source, destination, and the adjacent nodes to a current node in the routing, and that the ratio of the routing path length and the shortest Euclidean distance is bounded with a constant (competitive).

0: Set an initial planar triangulation.

1: Select a triangle at random and add a new node at the barycenter. Then, connect the new node to the three nodes of its triangle. Moreover, by iteratively applying diagonal flips, connect it to the nearest node (or more than one of the neighbor nodes) within a radius defined by the distance between the new node and the nearest node of its triangle.

2: The above process is repeated until reaching the required size N.

We have two versions with one nearest node and all neighbors in the circle. Note that these nodes are limited to the connected ones by applying iterative diagonal flips.

Fig. 1 shows the linking procedures by iterative diagonal flips: in a quadrilateral (of the shaded triangles) the diagonal link is exchanged to the other link (red line) for maximizing the minimum angle. The dashed lines are new links from the barycenter, and form new (five) triangles with the contours in the left of Fig. 1 (the intersected black solid links with dashed ones are removed). The difference for our model **based on the local procedures** is that diagonal flips in the original Delaunay triangulation are globally applied for the (entire) triangles until there exists no-increasing the minimum angle by the exchange of links.

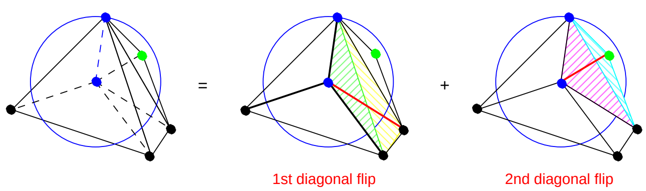

Figure 1: Linking procedures in a Delaunay-like SF net.

3. Simulation Results

Let us consider three classes of networks, RA: random Apollonian network, DT: Delaunay triangulation, and RA+NN: our Delaunay-like SF network model by the combination of random triangulation and diagonal flips to the ('one' or 'all') nearest neighbor(s) in the circle. Fig. 2 shows the topological characteristic that the RA+NN (the case of 'all' is the same with the 'one' property) has intermediate structure between those of RA and DT. Note that heterogeneous structures with dense and sparse parts are constructed. An explanation of the triangulation is the subdivision of a service area according to the increasing of population with preference of aggregation.

We discuss the details in numerical simulations. Each class of network is investigated in the averaging of 100 realizations at the size $N = 10,000$ generated from the initial configuration of a square graph at $(\pm 1, \pm 1)$ adding with the center at $(0, 0)$ and the four diagonal links. We have obtained similar results for the other initial configuration of triangle and hexagon.

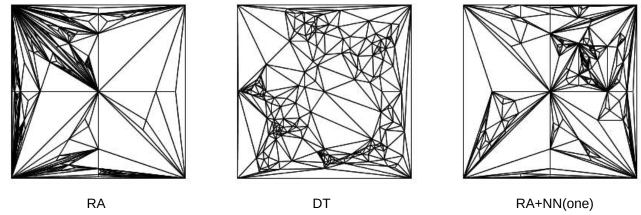

Figure 2: The three classes of networks, RA: random Apollonian network, DT: Delaunay triangulation, and RA+NN: our model by the combination of random triangulation and diagonal flips to the nearest neighbor.

model	estimated function	parameters
RA	$P(k) \sim k^{-\gamma_{RA}}$	$\gamma_{RA} \approx 3$
DT	$P(k) \sim \exp\left(-\frac{(\ln k - \mu)^2}{2\sigma^2}\right)$	$\mu = 1.7755$, $\sigma = 0.2383$
RA+NN(one)	$P(k) \sim k^{-\gamma} \exp(-ak)$	$\gamma = 2.26$, $a = 0.0647$
RA+NN(all)	$P(k) \sim k^{-\gamma} \exp(-ak)$	$\gamma = 1.7248$, $a = 0.0979$

Table 1: Estimated functions for the degree distributions by using the nonlinear MSE method.

Fig. 3 shows that the degree distributions in RA, RA+NN('one' & 'all'), and DT follow a power law, **power law with exponential cutoff**(see Appendix), and lognormal on the estimated dashed-lines whose parameters are summarized in Table 1. In other words, DT is not SF, while the other two models are so. We remark that in RA+NN('one' & 'all') the degrees of hubs become smaller than that in RA. It means **lower load or congestion at hubs**. The inset shows the degree-degree correlations; RA has a negative correlation, RA+NN('one' & 'all') have **more weaker ones**, while DT has a positive correlation. In general, the negative and positive correlations, characterized by connections between nodes with different (low and high) degrees and between nodes (such as hubs) with similar degrees, have been observed in technological or biological networks and in social networks [11], respectively. Thus, DT has a different topological structure (no-hubs, positive correlation) from others.

Next, we compare the communication costs. Fig. 4 (a) shows the average Euclidean distance $\langle D \rangle$ on the shortest paths between any nodes at the size N. The inset also shows the average Euclidean distance $\langle D' \rangle$ on the paths of the minimum hops at the size N. Note that the paths of the minimum hops may be different from the shortest paths. The distances $\langle D \rangle$ in DT and RA+NNs except in RA are smaller as the size N increases (dense network with many nodes), while the distances $\langle D' \rangle$ are the opposite. It is the reason for increasing the distances that the paths of the minimum hops tend to take long-range links. In particular, our pro-

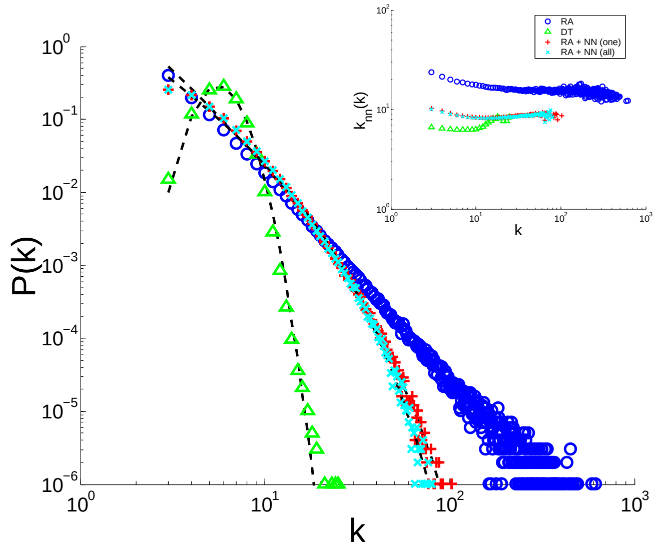

Figure 3: Distributions of degree $P(k)$ and degree-degree correlation $k_{nn}(k)$: Inset.

posed RA+NN('one') has the minimum distances of both $\langle D\rangle$ and $\langle D'\rangle$. From the estimated dashed lines (by using the MSE method), we obtain the relations $\langle D'\rangle \sim \log N^{\beta_d}$ and $\langle D'\rangle \sim \log N^{\beta'_d}$ characterized as a small-world effect.

Fig. 4 (b) shows the average minimum number of hops $\langle L\rangle$ at the size N. The inset also shows the average number of hops $\langle L'\rangle$ on the shortest paths in the Euclidean distance at the size N. All of them are smaller as the size N increases, RA+NNs have the intermediate values. The reason for larger $\langle L'\rangle$ than $\langle L\rangle$ is that the shortest paths tend to take short-range links in spite of increasing the number of hops such as on a zigzag route. From the estimated dashed lines, we obtain the relations $\langle L\rangle \sim N^{\alpha_l}$ and $\langle L'\rangle \sim N^{\alpha'_l}$. The estimated values of exponents are summarized in Table 2.

model	β_d	β'_d	α_l	α'_l	β_l	β'_l
RA	0.003	-0.015	0.121	0.136	0.554	0.814
DT	-0.013	0.155	0.333	0.455		
RA+NN(one)	-0.009	0.031	0.213	0.341	1.454	4.294
RA+NN(all)	-0.022	0.137	0.216	0.346	1.492	4.452

Table 2: Estimated values of the exponents.

4. Conclusion

In contrast to abstract graphs, many real networks are embedded in a metric space. On the other hand, the SF structure has been commonly found in many complex systems of biological, technological, and social origins [2]. It is therefore natural to investigate the possibility of embedding SF networks in space. **The topological and geographical properties are very important for efficient communication**.

In this paper, we have mentioned recent studies of geographical SF network models, and proposed a modified one to reduce long-range links. The Delaunay-like SF network is generated by the iterative triangulation and the diagonal flipping based on local rules, and embedded on a planar space without crossing links. Simulation results have shown that our proposed model has comparative **short links and small number of hops**. The statistical properties are inherited from the conventional Delaunay graphs [8] [9] and random Apollonian networks [5] [15], and suitable for many real systems such as distributed robots, sensor networks, and communication networks [4] [7], etc.

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Appendix

We approximately derive the exponential decaying in the degree distribution of our RAN+NN models. When some links are removed from a node by multiple diagonal flips as shown in Fig. 1 , the dynamical equation of the number of nodes $n(k+1, N)$ with degree $k+1$ at the size N is given

$$\begin{aligned} n(k+1, N+1) \quad = \quad & \tfrac{k}{N_\Delta} n(k, N) + \left(1 - \tfrac{k+1}{N_\Delta}\right) n(k+1, N) \\ & - a\tfrac{k}{N_\Delta} n(k+1, N), \end{aligned}$$

where N_Δ and a denotes the number of triangles and the average rate of the multiple diagonal flips, respectively. The 1st and 2nd terms in the r.h.s correspond to the preferential attachment (by random selection of triangles), and the 3rd term is the statistical effect of multiple diagonal flips. Note that there is no other reason for decreasing the degree. We neglect the other effects such as additional links to nodes with low degrees, because we focus on the tail of degree distribution.

By using $P(k) = n(k, N)/N$, we have

$$\frac{N_\Delta + N}{N} P(k+1) + k(P(k+1) - P(k)) + akP(k+1) = 0.$$

From the continuous approximation $dp/dk \approx P(k+1) - P(k)$ and $\gamma \stackrel{\text{def}}{=} (N_\Delta + N)/N$, it is rewritten as

$$k\frac{dp}{dk} = -(\gamma + ak)p.$$

Thus, we obtain the solution $p(k) \sim k^{-\gamma} \exp(-ak)$.

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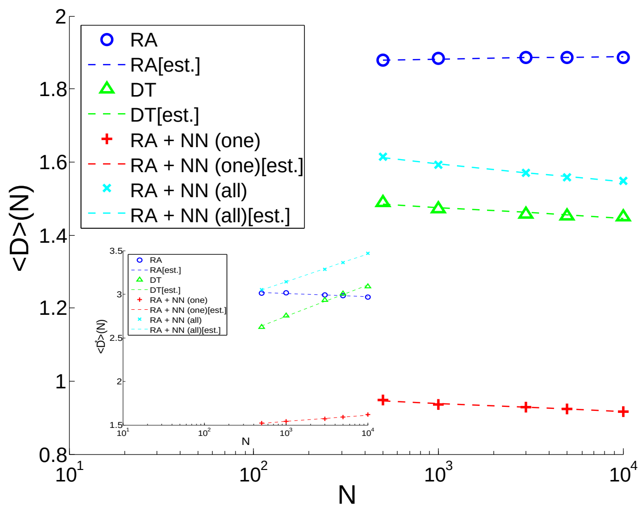
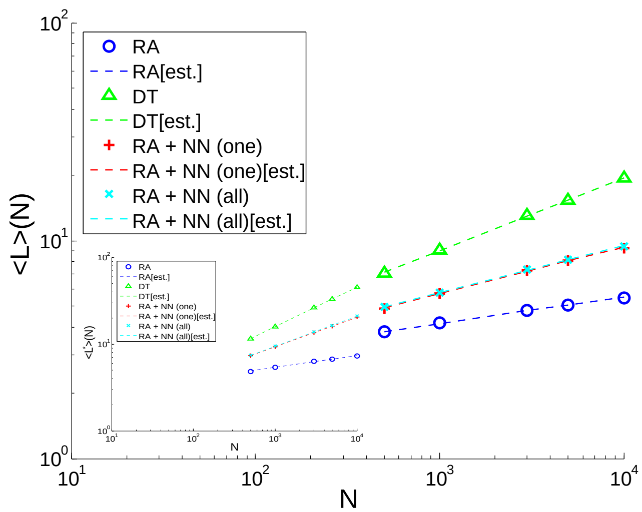


(a) $\langle D\rangle$ and $\langle D'\rangle$ vs. N in log-linear scale



(b) $\langle L\rangle$ and $\langle L'\rangle$ vs. N in log-log scale

Figure 4: Average distance $\langle D\rangle$ and minimum number of hops $\langle L\rangle$ between any nodes at the network size $N = 500, 1,000, 3,000, 5,000, 10,000$ from the initial triangulation of square. The dashed lines are the numerical estimations of $\langle D\rangle \sim \log N^{\beta_d}$ and $\langle L\rangle \sim N^{\alpha_l}$. Insets: the distance $\langle D'\rangle$ on the paths of minimum hops, and number of hops $\langle L'\rangle$ on the shortest paths. As shown in Table 2, we can also estimate other forms $\langle L\rangle \sim \log N^{\beta_l}$ and $\langle L'\rangle \sim \log N^{\beta_l'}$ except for DT. Note that DT is not the optimal triangulation in these criteria of the average distance and minimum hops.

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