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Global Solution to Nonholonomic System with Stochastic Feedbacks Based on Non-Smooth Stochastic Lyapunov Function

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Abstract—Stabilization problems of nonholonomic systems are generally difficult due to the lack of the existence of smooth state-feedback laws making the origins asymptotically stable. Recently, the problems are simplified by the strategy of adding white noises as parts of control inputs. Using the strategy, we tried further simplifications of control inputs and stability analysis by restricting the noises to be one-dimensional and using the approach of non-smooth stochastic Lyapunov functions (SLFs). However, the analysis is still under construction because the SLFs do not ensure the existence of global solutions. In this paper, we carry on the analysis by providing the global solutions.

1. Introduction

Nonholonomic systems are mechanical systems with constraints on their velocity that are not derivable from position constraints [1]. Many control systems such as cars, underwater vehicles, snake robots, and space robots are included in a class of nonholonomic systems. Hence, control problems of nonholonomic systems are important for the basic development of control engineering. However, controlling nonholonomic systems are difficult because many of them have no continuous state-feedback stabilizer [2]. Therefore, discontinuous state-feedback laws [3] or time varying state-feedback laws [4] are considered while the designs of them generally complicated.

Recently, the effective use of stabilization by noise is proposed for deriving a simple approach to design stabilizers of nonholonomic systems [5]. This strategy provides state-feedback laws with Gaussian white noises such that the states of the target system converges to the origin with probability one. Because disturbance terms exists, the resulting stochastic systems are represented by stochastic differential equations. The stability of the systems are analyzed via stochastic Lyapunov theory. The characteristic feature of the analysis to deal with the systems like time-independent systems while they are, in fact, time-varying.

Using the above strategy of stabilization by noise, we tried to stabilize a chained system with just a one-dimensional Wiener process [6]. This provides stochastic control laws simpler than previous works in [5] and [7]. While we designed a non-smooth stochastic Lyapunov function (SLF) for showing the stabilization, the proof is under construction due to the lack of ensuring global solutions to the stochastic differential equation and analyzing the behavior of the states in the region that the SLF is non-smooth. In this paper, we show the existence of global solution to the closed-loop system.

Notations. $\mathbb{R}^n$ denotes an n-dimensional Euclidean space. The conditional probability of some event A, under the condition B, is represented by $\mathbb{P}\{A|B\}$. Function $w_1 \in \mathbb{R}$ denotes a one-dimensional Wiener process. For $k_1, k_2 : \mathbb{R}^n \to \mathbb{R}$, $(L_{k_1}k_2)(x) := (\partial k_2/\partial x)k_1(x)$. For a real number x, $\mathrm{sgn}(x)$ denotes 1 if $x > 0$, 0 if $x = 0$, and -1 if $x < 0$.

2. Stabilization by Artificial Wiener Processes

In this section, we consider stochastic differential equations appropriate for our problem formulation.

Gaussian white noises that we add to control inputs are generated by using Wiener processes. However, the indifferentiability of Wiener processes prevents us from generating Gaussian white noises because the indifferentiability implies, roughly, Wiener processes have infinity-large values as their derivatives. To consider actually-generable white noises, we take the notion of artificial Wiener processes [5].

In this paper, we consider the Wong-Zakai-type artificial Wiener process $w^D(t)$ shown in Fig. 1; it is linear-interpolated discrete Wiener process $\{w^D(t_0), w^D(t_1), ..., w^D(t_n)\}$:

$$w^D(t) = \frac{w^D(t_k) - w^D(t_{k-1})}{t_k - t_{k-1}}(t - t_{k-1}), t \in [t_{k-1}, t_k]. \quad (1)$$

As $N \to \infty$, the solution to a system

$$\dot{x} = f(x) + \sigma(x)\dot{w}^D, x \in \mathbb{R}^n, f, \sigma : \mathbb{R}^n \to \mathbb{R}^n, \quad (2)$$

converges to the solution to

$$dx = \left(f(x) + f(x)^{WZ}\right)dt + \sigma(x)dw, \quad (3)$$

where

$$f^{WZ}(x) = \frac{1}{2}\frac{\partial\sigma(x)}{\partial x}\sigma(x) \quad (4)$$

is said to be a Wong-Zakai correction term [5].

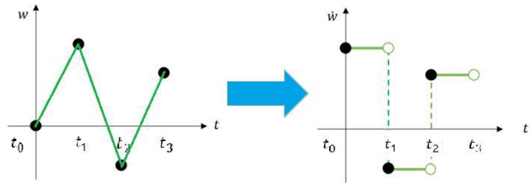

Figure 1: Wong-Zakai-type artificial Wiener process

3. Stochastic Lypunov Theory

In this section, Lyapunov theory of stochastic systems and stability of stochastic systems are explained.

Consider (3) with the assumption of $f(0) = 0$ and $\sigma(0) = 0$. In this section, we assume the existence of a global solution for (3).

Definition 1 ([9]) *The equilibrium $x(t) \equiv 0$ of the system (3) is globally asymptotically stable in probability if*

$$\lim_{x(0)\to 0} \mathbb{P}\left\{\sup_{0\le t}(x(t)) > \epsilon\right\} = 0 \tag{5}$$

holds for any $\epsilon > 0$, and

$$\mathbb{P}\left\{\lim_{t\to+\infty}|x(t)| = 0\right\} = 1 \tag{6}$$

for any $x(0) \in \mathbb{R}^n$. □

We also define an infinitesimal generator of $v : \mathbb{R}^n \to \mathbb{R}$ for (3) by

$$(\mathcal{L}v)(x) = (L_f v)(x) + \frac{1}{2}(L_\sigma(L_\sigma v))(x). \tag{7}$$

Definition 2 ([9]) *The function $v : \mathbb{R}^n \to \mathbb{R}$ is a stochastic Lyapunov function (SLF) if $v : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable in $x \in \mathbb{R}^n$, radially unbounded in $x \in \mathbb{R}^n$, and $(\mathcal{L}v)(x(t))$ is negative definite for all $x \in \mathbb{R}^n$.* □

Using the above definitions, the following is obtained.

Theorem 1 ([9]) *The equilibrium $x(t) \equiv 0$ of the system (3) is globally asymptotically stable in probability if there exists an SLF.* ♦

4. Stabilization of Nonholonomic System by Noise

In this paper, we consider a nonholonomic system

$$\dot{x} = g_1(x)u_{c1} + g_2(x)u_{c2} = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \\ b_3x_2 & -b_4x_1 \end{bmatrix}\begin{bmatrix} u_{c1} \\ u_{c2} \end{bmatrix}, \tag{8}$$

where $b_1, b_2, b_3, b_4 > 0$ and $b_2b_3 - b_1b_4 \neq 0$. Further, consider state-feedback with stochastic terms

$$u_i = v_i(x) + c_i(x)\dot{w}^D, \tag{9}$$

where $i = 1, 2$. Then, system (8) becomes (3) with

$$f(x) = \sum_{i=1}^{2} g_i(x)v_i(x), \sigma(x) = \sum_{i=1}^{2} g_i(x)c_i(x). \tag{10}$$

Thus, our aim is to design v_1, v_2, c_1 and c_2 such that the origin of (3) becomes globally asymptotically stable in probability.

Consider feedback laws of the form:

$$v_1(x) = \alpha_1 x_1 + \beta_1(x)x_2 - d(x), \tag{11}$$

$$v_2(x) = \alpha_2 x_2 + \beta_2(x)x_1 + d(x)\mathrm{sgn}(Bx_3), \tag{12}$$

$$B := \frac{1}{b_2b_3 - b_1b_4}, \tag{13}$$

and diffusion coefficients

$$c_1(x) = -|Bx_3|^{\frac{1}{2}}, \tag{14}$$

$$c_2(x) = |Bx_3|^{\frac{1}{2}}\mathrm{sgn}(Bx_3), \tag{15}$$

where α_1 and α_2 are designed to satisfy the following conditions:

$$\alpha_1, \alpha_2 < 0, \quad b_3\alpha_1 - b_4\alpha_2 > 0, \tag{16}$$

$$(b_3\alpha_1 - b_4\alpha_2) + 2Bb_3b_4 > 0. \tag{17}$$

The other functions are designed as follows:

$$\beta_1(x) = \frac{\alpha_2 b_4}{k_1 b_1}k_3\mathrm{sgn}(x_3), \beta_2(x) = -\frac{\alpha_1 b_3}{k_2 b_2}k_3\mathrm{sgn}(x_3), \tag{18}$$

$$d(x) = \frac{1}{4}\mathrm{sgn}(Bx_3)\{b_3x_2 + b_4x_1\mathrm{sgn}(Bx_3)\}. \tag{19}$$

Thus, the resulting system is

$$dx = F(x)dt + G(x)dw, \tag{20}$$

where

$$F(x) = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2}x_3(t) \end{bmatrix} + g_1(x)(v_1(x) + d(x)) + g_2(x)(v_2(x) - d(x)\mathrm{sgn}(Bx_3)),$$

$$G(x) = g_1c_1(x) + g_2c_2(x). \tag{21}$$

A candidate of an SLF is provided as follows:

$$V_0(x) = v_0(x) + k_3|x_3(t)|, v_0(x) = \frac{k_1}{2}x_1(t)^2 + \frac{k_2}{2}x_2(t)^2. \tag{22}$$

Then, we obtain following:

Theorem 2 ([6]) *Let us consider* (20). *An infinitesimal generator of* (22) *becomes $\mathcal{L}V_0 < 0$ for all $x \in \mathbb{R}^3 \setminus M :=$* $\{x \in \mathbb{R}^3 | x_3 = 0\}$. ♦

From Theorems 1 and 2, a system (3) is expected to be globally asymptotically stable in probability because (22) becomes an SLF. However, the stability analysis has not been completed because $(\mathcal{L}V_0)(x)$ is impossible to be definite on M. Furthermore, before considering stability, we have to confirm that (20) has a global solution.

5. Existence of Global Solution

In this section, we show the existence of a global solution to (20).

Definition 3 ([8]) *System (3) is said to be forward complete in probability (FCiP) if for each $x_0 \in \mathbb{R}^n$, there exists a continuous function $\psi : [0,\infty)\times(0,1) \to [0,\infty)$ such that*

$$\mathbb{P}[\forall t \in [0,\infty), |x(t)| \le \psi(t,\epsilon)] \ge 1-\epsilon \tag{23}$$

holds for any $\epsilon \in (0,1)$ □

Roughly speaking, FCiP ensures that sample paths of $x(t)$ exists for any large time value $t \in [0,\infty)$. Therefore, we conclude that there exists a global solution to (20) if the system is FCiP. A useful sufficient condition for FCiP is as follows:

Theorem 3 ([8]) *Suppose that system (3) admits a positive definite, proper C^2 function $Y : \mathbb{R}^n \to \mathbb{R}$ satisfying*

$$(\mathcal{L}Y)(x) \le cY(x) + d \tag{24}$$

for all $x \in \mathbb{R}^n$ and for some constants $c, d \in [0,\infty)$. Then system (3) is FCiP. ♦

Then, we state our main claim.

Theorem 4 *The system (20) is FCiP.* □

To prove Theorem 4, we set

$$Y = \begin{cases} V_0(x), & b < |x_3(t)| \\ V_2(x) = v_0(x) + V_{12}(x_3), & a \le |x_3(t)| \le b \\ V_1(x) = v_0(x) + \dfrac{k_3}{2}x_3(t)^2, & |x_3(t)| < a \end{cases}, \tag{25}$$

for $0 < a < b$, where

$$V_{12}(x_3) = a_5|x_3^5| + a_4x_3^4 + a_3|x_3^3| + a_2x_3^2 + a_1|x_3| + a_0, \tag{26}$$

$$a_0 = -\frac{3a^3b^3(a+b-4)}{2(a-b)^5}, \tag{27}$$

$$a_1 = \frac{a^2}{2(a-b)^5}\left\{2a^3 + a^2b(9b-10) + 4ab^2(3b-4) + 9(b-4)b^3\right\}, \tag{28}$$

$$a_2 = -\frac{b}{2(a-b)^5}\left\{9a^4 + 18a^3(b-2) + 4a^2b(7b-12) + 4a(b-9)b^2 + b^4\right\}, \tag{29}$$

$$a_3 = \frac{3}{2(a-b)^5}\left\{a^4 + 4a^3(b-1) + 2a^2b(5b-8) + 4a(b-4)b^2 + (b-4)b^3\right\}, \tag{30}$$

$$a_4 = \frac{-3a^3 + 4a^2(4-3b) + 4a(7-3b)b + (16-3b)b^2}{2(a-b)^5}, \tag{31}$$

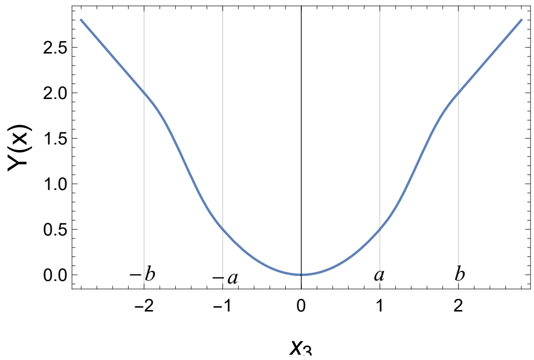

Figure 2: An example of $Y(x)$ with $a = 1$ and $b = 2$.

$$a_5 = \frac{a^2 + a(4b-6) + (b-6)b}{2(a-b)^5}. \tag{32}$$

The function (26) connects V_0 and V_1 so that Y is C^2 for all $x \in \mathbb{R}^n$; see Fig. 2. If $b < |x_3|$, $\mathcal{L}Y(x) = \mathcal{L}V_0(x) < 0$ via Theorem 2. Therefore, to prove Theorem 4, we consider the other two situations of $Y = V_1$ and $Y = V_2$:

Lemma 1 *There exist c and d satisfying (24) for each x_3 satisfying $|x_3| < a$.* ♦

Lemma 2 *There exist c and d satisfying (24) for each x_3 satisfying $a \le |x_3| \le b$.* ♦

If the above two lemmas are true, then we conclude that our main result of Theorem 4 is true. The rest of this section provides the brief sketches of the proofs of the lemmas.

5.1. The Proof of Lemma 1

The brief sketch of the proof is as follows.
i) If $x_3 = 0$,

$$(\mathcal{L}V_1)(x) = b_1k_1\alpha_1x_1^2 + b_2k_2\alpha_2x_2^2 \le 0 \tag{33}$$

because $\alpha_1, \alpha_2 < 0$.
ii) If $0 < |x_3| < a$,

$$(\mathcal{L}V_1)(x) \le \gamma_1x_1^2 + \gamma_2x_2^2 + \gamma_3|x_1x_2| + \gamma_4, \tag{34}$$

where

$$\gamma_1 = \frac{|B|b_4^2k_3a}{2}, \gamma_2 = \frac{|B|b_3^2k_3a}{2}, \tag{35}$$

$$\gamma_3 = (a+1)(b_3\alpha_1 - b_4\alpha_2) + 2Bb_3b_4a, \tag{36}$$

$$\gamma_4 = \frac{1}{2}\left\{b_1^2k_1 + b_2^2k_2\right\}|B|a. \tag{37}$$

Using $2\gamma_3|x_1x_2| \le x_1^2 + \gamma_3^2 x_2^2$, we obtain

$$\begin{aligned}
(\mathcal{L}V_1)(x) &\le \gamma_1 x_1^2 + \gamma_2 x_2^2 + \frac{x_1^2}{2} + \gamma_3^2\frac{x_2^2}{2} + \gamma_4 \\
&\quad + \frac{1}{k_2}(2\gamma_2 + \gamma_3^2)\left(\frac{k_1}{2}x_1^2 + \frac{k_3}{2}x_3(t)^2\right) \\
&\quad + \frac{1}{k_1}(2\gamma_1 + 1)\left(\frac{k_2}{2}x_2^2 + \frac{k_3}{2}x_3(t)^2\right) \\
&= cV_1 + d, \qquad (38)
\end{aligned}$$

where

$$c = \frac{2}{k_1}\gamma_1 + \frac{2}{k_2}\gamma_2 + \frac{1}{k_2}\gamma_3^2 + \frac{1}{k_1}, d = \gamma_4. \qquad (39)$$

Thus, considering the combination of (33) and (38), we conclude that Lemma 1 is true.

5.2. The Proof of Lemma 2

The brief sketch of the proof is as follows.

An infinitesimal generator of $V_2(x)$ is calculated as the following:

$$(\mathcal{L}V_2)(x) \le \gamma'_1 x_1^2 + \gamma'_2 x_2^2 + \gamma'_3|x_1x_2| + \gamma'_4 \qquad (40)$$

where,

$$\gamma'_1 = -\frac{b_3b_4k_3}{b_1b_2k_1k_2}Cb_1k_1\alpha_1 + \frac{|B|b_4^2 b}{2}D, \qquad (41)$$

$$\gamma'_2 = -\frac{b_3b_4k_3}{b_1b_2k_1k_2}Cb_2k_2\alpha_1 + \frac{|B|b_3^2 b}{2}D, \qquad (42)$$

$$\gamma'_3 = (b_4\alpha_2 - b_3\alpha_1)(k_3 + C) + \left\{(Bb_3b_4b)^2 + 1\right\}D, \qquad (43)$$

$$\gamma'_4 = \frac{b}{2}\left\{B(b_1^2k_1 + b_2^2k_2) + C\right\}, \qquad (44)$$

$$\begin{aligned}
C &= (a_1^2 + 1) + 2(a_2^2 + 1)b + 3(a_3^2 + 1)b^2 \\
&\quad + 4(a_4^2 + 1)b^3 + 5(a_5^2 + 1)b^4, \qquad (45)
\end{aligned}$$

$$\begin{aligned}
D &= 2(a_2^2 + 1) + 6(a_3^2 + 1)b \\
&\quad + 12(a_4^2 + 1)b^2 + 20(a_5^2 + 1)b^3. \qquad (46)
\end{aligned}$$

Using $2\gamma'_3|x_1x_2| \le x_1^2 + \gamma_3'^2 x_2^2$, we obtain

$$\begin{aligned}
(\mathcal{L}V_2)(x) &\le \gamma'_1 x_1^2 + \gamma'_2 x_2^2 + \frac{x_1^2}{2} + \gamma_3'^2\frac{x_2^2}{2} + \gamma'_4 \\
&\quad + \frac{1}{k_2}(2\gamma'_2 + \gamma_3'^2)\left(\frac{k_1}{2}x_1^2 + \frac{k_3}{2}x_3(t)^2\right) \\
&\quad + \frac{1}{k_1}(2\gamma'_1 + 1)\left(\frac{k_2}{2}x_2^2 + \frac{k_3}{2}x_3(t)^2\right) \\
&= cV_2 + d, \qquad (47)
\end{aligned}$$

where

$$c = \frac{2}{k_1}\gamma'_1 + \frac{2}{k_2}\gamma'_2 + \frac{1}{k_2}\gamma_3'^2 + \frac{1}{k_1}, d = \gamma'_4. \qquad (48)$$

Therefore, we conclude that Lemma 2 is true.

6. Concluding Remarks

In this paper, we show the existence of a global solution to a randomized nonholonomic system having a non-smooth stochastic Lyapunov function. The rest problem is to finish the confirmation of the global asymptotic stability in probability of the origin. We plan to complete it by further investigation of the properties of the system derived from the combination of the results of this paper and the non-smooth stochastic Lyapunov function approach in our previous work [6].

Acknowledgments

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