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Controlling Hyperchaotic n×m-Scroll Attractors

Guo-Ping Jiang†, Wallace Kit-Sang Tang‡, and Guanrong Chen‡

† Department of Electronic Engineering, Nanjing University of Posts & Telecommunications
Nanjing, 210003, China
‡ Department of Electronic Engineering, City University of Hong Kong, Hong Kong SAR, China
Email: jianggp@njupt.edu.cn, kstang@ee.cityu.edu.hk, gchen@ee.cityu.edu.hk

Abstract–In this paper, a simple output-feedback control scheme is presented for stabilizing a hyperchaotic system with $n\times m$-scrolls. Using only two state variables, the hyperchaotic system can be stabilized at any desired equilibrium of the saddle type. The effectiveness of the proposed method is well demonstrated by simulations.

1. Introduction

Over the past decade, the fundamental issue of generating complex attractors has raised a lot of interests. One of the major classes of complex attractors is the so-called hyperchaos, which has more than one positive Lyapunov exponent, meaning that the dynamics of such a system expand in more than one direction therefore a more complex attractor can be obtained.

Due to its great potential in technological applications, different approaches have been proposed for the generation of hyperchaos [1-5]. Although quite many of them are obtained from smooth systems based on trial-and-error, it is also possible to obtain hyperchaotic dynamics by coupling two or more regular chaotic systems. Cafagna and Grassi [2] demonstrated that $n\times m$-scroll attractors can be generated from two coupled modified Chua's circuits with the sine-nonlinearity [6].

To further facilitate the use of hyperchaos, it is equally important to control the hyperchaotic attractor or to stabilize the hyperchaotic system into its equilibrium points. In the past, a general approach is based on a proportional state-feedback control where, however, precise knowledge of the target orbit in the phase space is needed. This makes the approach questionable for chaotic systems, where the target equilibrium points are generally dependent on some precise values of system parameters which are usually unknown or very sensitive.

Recently, in [7] and [8], it was suggested to use the classical derivative control method for stabilizing unstable equilibrium points of some (hyper)chaotic systems. The fundamental limitation of this approach is that some unstable equilibrium points cannot be stabilized. As demonstrated in [7] with an $n\times m$-scroll attractor, the equilibrium point can be stabilized by this method, provided that the real parts of all its four complex eigenvalues are positive. Moreover, this approach is very sensitive to higher-frequency fluctuations [9] because of the use of the derivatives of the system state variables.

Controllers using the conventional low-pass filters have also been proposed in [9] to stabilize unstable equilibrium points of some dynamical systems. By estimating the location of the target equilibrium point from the filtered DC output signal of the system, the difference between the actual and the filtered output signals is used as the control signal. However, as is well known, this method is not applicable if there is an odd number of real positive eigenvalues in the Jacobian of the linearized controlled system.

In this paper, an output-feedback controller based on a multivariable system's state-observer approach [10] is used for stabilizing all kinds of unstable equilibrium points of a hyperchaotic system with $n\times m$-scroll attractors, generated from two coupled modified Chua's circuits with the sine-nonlinearity. The design of the controller is simple and it only depends on partial system state variables expressed in a simple linear combination form. However, the controller is very effective, as will be demonstrated by simulations in Section 4 of the paper.

2. Hyperchaotic $n\times m$-Scroll Attractors

The dimensionless equations of the hyperchaotic Chua's circuit with $n\times m$-scroll attractors [2, 7] can be expressed as

$$\begin{aligned}
\dot{x}_1 &= a\,(x_2 - f_1(x_1)) \\
\dot{x}_2 &= x_1 - x_2 + x_3 + m_1(x_5 - x_2) \\
\dot{x}_3 &= -b\,x_2 \\
\dot{x}_4 &= a\,(x_5 - f_2(x_4)) \\
\dot{x}_5 &= x_4 - x_5 + x_6 + m_2(x_2 - x_5) \\
\dot{x}_6 &= -b\,x_5
\end{aligned} \tag{1}$$

where m_1 and m_2 are coupling factors; $f_1(\cdot)$ and $f_2(\cdot)$ are smooth sine-type functions express as [2, 6, 7]:

$$f(x)=\begin{cases}\dfrac{bp}{2a}(x-2ac) & \text{if } x\ge 2ac\\ -b\sin(\dfrac{px}{2a}+d) & \text{if } -2ac<x<2ac\\ \dfrac{bp}{2a}(x+2ac) & \text{if } x\le -2ac\end{cases}$$

Figure 1 shows a typical hyperchaotic 3×4-scroll attractor obtained by simulation while Fig. 2 shows results from an electronic circuit based on the design in [6].

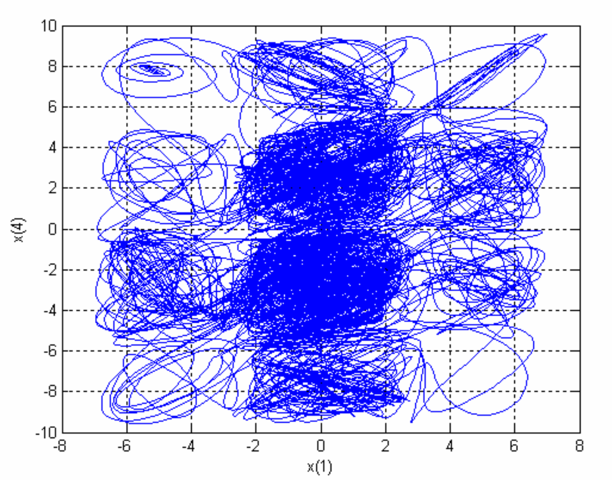


Fig. 1. Hyperchaotic 3×4-scroll attractor: $a = 10.814$, $b = 14$, $m_1 = 0.25$, $m_2 = 0.25$, and $a = 1.3$, $b = 0.11$, $c_1 = 2$, $d_1 = p$, $c_2 = 3$, $d_2 = 0$ for the nonlinear functions $f_1(\cdot)$ and $f_2(\cdot)$.

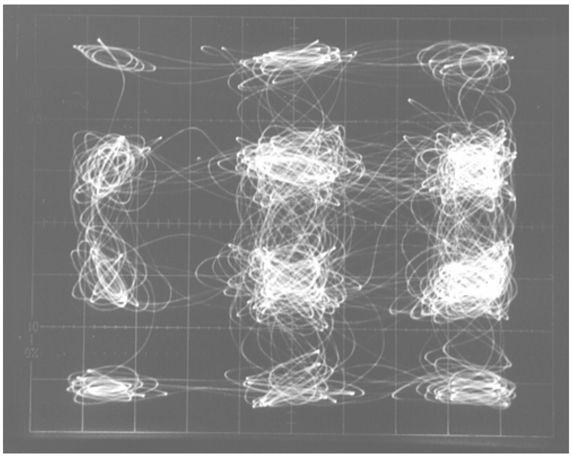


Fig. 2. Hyperchaotic 3×4-scroll attractor obtained from electronic circuit.

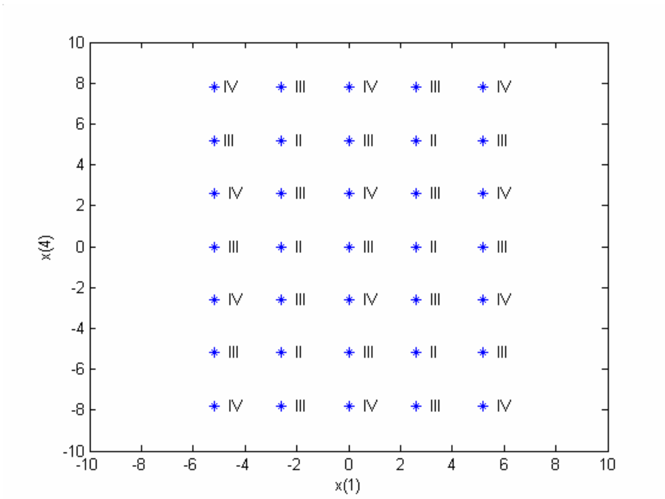


Fig. 3. Locations of the equilibrium points of the hyperchaotic system in Fig. 1.

Here, the positions of the equilibrium points of this hyperchaotic system can be exactly computed by combining the equilibrium points $(x_{1eq}, 0, -x_{1eq})$ of the first modified Chua's circuit (1st subsystem) and the equilibrium points $(x_{4eq}, 0, -x_{4eq})$ of the second modified Chua's circuit (2nd subsystem). They possess a generic form with $(x_{1eq}, 0, -x_{1eq}, x_{4eq}, 0, -x_{4eq}) \in \mathbf{R}^6$. An example is depicted in Fig. 3, where the 35 equilibrium points of the hyperchaotic system in Fig. 1 are located.

According to [2, 7], these equilibrium points can be classified as follows:

1) *Saddle type-II:* There are only two positive real eigenvalues, with $f_1'(x_{1eq}) = -\frac{bp}{2a} < 0$ and $f_2'(x_{4eq}) = -\frac{bp}{2a} < 0$.

2) *Saddle type-III:* It is characterized by one real and two complex eigenvalues with positive real parts. In this case, $f_1'(x_{1eq}) = -\frac{bp}{2a} < 0$ and $f_2'(x_{4eq}) = \frac{bp}{2a} > 0$, or $f_1'(x_{1eq}) = \frac{bp}{2a} > 0$ and $f_2'(x_{4eq}) = -\frac{bp}{2a} < 0$.

3) *Saddle type-IV:* It consists of four complex eigenvalues, all with positive real parts. In this case, $f_1'(x_{1eq}) = \frac{bp}{2a} > 0$ and $f_2'(x_{4eq}) = \frac{bp}{2a} > 0$.

3. Controlling Hyperchaotic $n\times m$-Scroll Attractors

Here, the hyperchaotic system with $n\times m$-scroll attractors (1) is to be stabilized by using output-feedback based on the state-observer approach [10]. From (1) and let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}, \quad \mathbf{g}(\mathbf{x}) = \begin{bmatrix} -a\ f_1(x_{1eq}) \\ 0 \\ 0 \\ -a\ f_2(x_{4eq}) \\ 0 \\ 0 \end{bmatrix}, \text{ and}$$

$$\mathbf{A} = \begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 \\ 1 & -1-m_1 & 1 & 0 & m_1 & 0 \\ 0 & -b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ 0 & m_2 & 0 & 1 & -1-m_2 & 1 \\ 0 & 0 & 0 & 0 & -b & 0 \end{bmatrix}.$$

Then, the controlled system becomes [10]

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x}) + l\ \mathbf{L}(\mathbf{y} - \mathbf{y}_s) \qquad (2)$$

where $\mathbf{y} = \mathbf{Cx}$ is the output of system (1) with a constant matrix $\mathbf{C} \in \mathbf{R}^{p\times 6}$, $\mathbf{y}_s = \mathbf{Cx}_s$ is the observation of the targeted equilibrium $\mathbf{x_s}$, $\mathbf{L} \in \mathbf{R}^{6\times p}$ is the control gain matrix, and $l\,\mathbf{L}(\mathbf{y}-\mathbf{y_s})$ is considered as the output feedback controller with

$$l = \begin{cases} 1, & \text{if } \mathbf{x} \in \Omega_{\mathbf{x_s}} \\ 0, & \text{else} \end{cases} \tag{3}$$

in which $\Omega_{\mathbf{x_s}}$ denotes a (small) neighborhood of the targeted equilibrium point $\mathbf{x_s}$ and can be determined as $\Omega_{\mathbf{x_s}} = \{\mathbf{x} : \|\mathbf{x}-\mathbf{x_s}\| \le e\}$, where e is a (small) positive constant. If the trajectory is within $\Omega_{\mathbf{x_s}}$, it is said to be close to the target equilibrium point $\mathbf{x_s}$.

The Jacobian $\mathbf{J}$ of the controlled system (2) at the equilibrium point $(x_{1eq}, 0, -x_{1eq}, x_{4eq}, 0, -x_{4eq})$ is

$$\begin{aligned} \mathbf{J} &= (\mathbf{A}+\mathbf{LC}+\mathbf{g}'(\mathbf{x_s})) \\ &= (\overline{\mathbf{A}}+\mathbf{LC}) \end{aligned} \tag{4}$$

where $\overline{\mathbf{A}} = \mathbf{A}+\mathbf{g}'(\mathbf{x_s})$ or

$$\overline{\mathbf{A}} = \begin{bmatrix} -a\,f_1'(x_{1eq}) & a & 0 & 0 & 0 & 0 \\ 1 & -1-m_1 & 1 & 0 & m_1 & 0 \\ 0 & -b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a\,f_2'(x_{4eq}) & a & 0 \\ 0 & m_2 & 0 & 1 & -1-m_2 & 1 \\ 0 & 0 & 0 & 0 & -b & 0 \end{bmatrix} \tag{5}$$

Choose $\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. Then, $(\overline{\mathbf{A}}, \mathbf{C})$ will be observable at any equilibrium point, and the matrix $\mathbf{L} = \begin{bmatrix} l_{11} & l_{21} & l_{31} & l_{41} & l_{51} & l_{61} \\ l_{12} & l_{22} & l_{32} & l_{42} & l_{52} & l_{62} \end{bmatrix}^T$ can be found by pole placement technique, so as to stabilize the controlled hyperchaotic system (2).

With our choice of $\mathbf{C}$, $\mathbf{y_s} = \mathbf{Cx_s} = \mathbf{0}$ and the controller becomes $\mathbf{u} = l\,\mathbf{Ly}$, so that the controlled system becomes

$$\begin{aligned} \dot{x}_1 &= a\,(x_2 - f_1(x_1)) + l\,(l_{11}x_2 + l_{12}x_5) \\ \dot{x}_2 &= x_1 - x_2 + x_3 + m_1(x_5 - x_2) + l\,(l_{21}x_2 + l_{22}x_5) \\ \dot{x}_3 &= -b\,x_2 + l\,(l_{31}x_2 + l_{32}x_5) \\ \dot{x}_4 &= a\,(x_5 - f_2(x_4)) + l\,(l_{41}x_2 + l_{42}x_5) \\ \dot{x}_5 &= x_4 - x_5 + x_6 + m_2(x_2 - x_5) + l\,(l_{51}x_2 + l_{52}x_5) \\ \dot{x}_6 &= -b\,x_5 + l\,(l_{61}x_2 + l_{62}x_5) \end{aligned} \tag{6}$$

From (6), it can be noticed that no precise value of the targeted equilibrium point $(x_{1eq}, 0, -x_{1eq}, x_{4eq}, 0, -x_{4eq})$ is needed.

4. Simulation Results

Due to the space limitation, we only focus on the stabilization problem for the saddle type-II equilibrium points, for which the existing control techniques [7,8] fail to succeed.

Taking the hyperchaotic system (1) in Fig. 1 as an illustrative example, from (5), we have

$$\overline{\mathbf{A}} = \begin{bmatrix} 1.4373 & 10.814 & 0 & 0 & 0 & 0 \\ 1 & -1.25 & 1 & 0 & 0.25 & 0 \\ 0 & -14 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.4373 & 10.814 & 0 \\ 0 & 0.25 & 0 & 1 & -1.25 & 1 \\ 0 & 0 & 0 & 0 & -14 & 0 \end{bmatrix}.$$

By simple calculation, we obtain

$$\mathbf{L} = \begin{bmatrix} 32.4599 & 4.1479 & -18.8191 & -0.2823 & 0.1351 & 0.5980 \\ -0.7747 & 0.0661 & 0.8145 & 34.0582 & 4.3267 & -19.5003 \end{bmatrix}^T$$

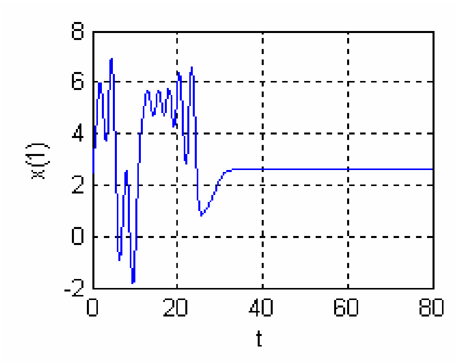
and the eigenvalues of the Jacobian $\mathbf{J} = (\overline{\mathbf{A}}+\mathbf{LC})$ are –1.2±1.8j, –1.5±2.2j, –1.8, and -0.9, all lying on the left-half plane.

The controlled system can now be stabilized at the desired saddle type-II equilibrium points, as shown in Figs. 4 and 5, by turning on the controller at different instants. For example, in Fig. 4, the controller is turned on at $t_s = 25s$, while the trajectory comes close to the equilibrium point $(2.6, 0, -2.6, 0, 0, 0)$. It can be observed that the system is stabilized to the corresponding equilibrium point. For the case in Fig. 5, the controller is turned on at $t_s = 40s$ in order to stabilize the system to the equilibrium point $(-2.6, 0, 2.6, 0, 0, 0)$.

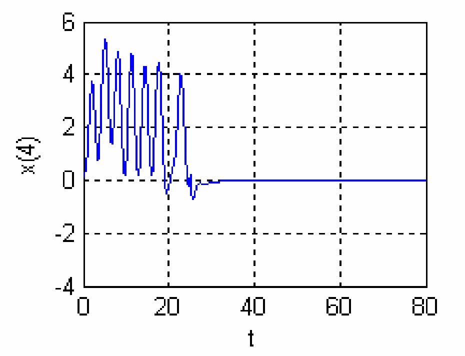
It should be noted that the same controller is used for both cases. Similarly, other saddle type-II equilibrium points can also be stabilized if the controller is started at suitable times when the hyperchaotic trajectory comes close to the targeted equilibrium point.

5. Conclusions

In this paper, an output-feedback controller has been designed to stabilize the unstable equilibrium points of a hyperchaotic system with $n\times m$ -scroll attractors, generated from two weakly-coupled modified Chua's circuits with the sine-nonlinearity. Based on Jacobian analysis and the pole placement technique, the control parameters can be duly obtained. Simulations has illustrated that the hyperchaotic system with $n\times m$ -scroll attractors can be stabilized to equilibrium points of saddle type-II, for which the existing control techniques fail to succeed. It should be pointed out that the developed method can also be applied to many other hyperchaotic systems and for different types of target equilibrium points.

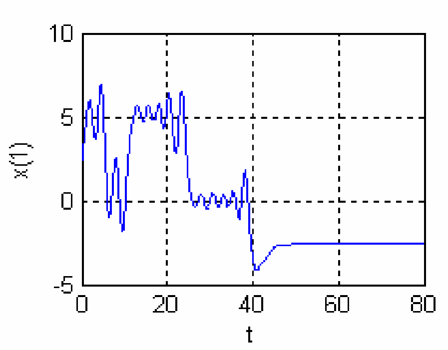


(a) $x_1(t)$

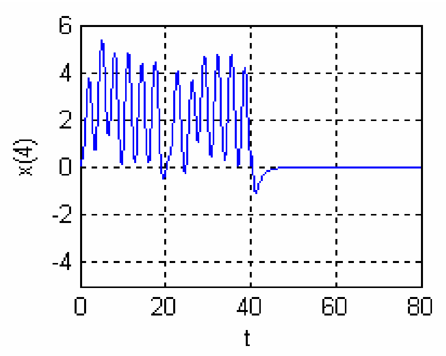


(b) $x_4(t)$

Fig. 4. Stabilizing hyperchaotic system to saddle type-II equilibrium point (2.6, 0, -2.6, 0, 0, 0).



(a) $x_1(t)$



(b) $x_4(t)$

Fig. 5. Stabilizing hyperchaotic system to saddle type-II equilibrium point (-2.6, 0, 2.6, 0, 0, 0).

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