

Feedback Control to Avoid Time-delay Induced Amplitude Death

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Abstract—The time-delay induced amplitude death has created considerable interest in the field of nonlinear science. From the viewpoint of engineering applications which employ oscillatory behavior in coupled oscillators, death phenomenon should be avoided. The present paper provides a systematic procedure how to design a feedback controller which never induces amplitude death. The controller, which is added to each individual oscillator, is designed on the basis of the odd number property derived in [Konishi, Phys. Rev. E, **67** (2003) 017201].

1. Introduction

The diffusive coupled nonidentical oscillators can cease their oscillations. This phenomenon, that is called *Amplitude Death* or *Oscillation Death*, has been investigated [1, 2, 3, 4, 5]. It was clarified that death phenomenon never occurs in diffusive coupled identical oscillators [2, 4, 6, 7]. On the other hand, Reddy *et al.* [8] showed that death phenomenon can occur when the identical oscillators coupled by *time delay* connection. This time-delay induced amplitude death has created considerable interest in the field of nonlinear science [9]. The death was observed in electrical circuits [10] and thermo-optical oscillators [11], and was theoretically studied as follows: the stability of death in coupled simple oscillators near Hopf bifurcations [12], the occurrence of death in identical oscillators with dynamical connections [13], and the sufficient condition (odd number property) under which the death never occurs in coupled oscillators with dynamical connection [13] and with time-delay connection [6, 7].

From the viewpoint of engineering applications which employ oscillatory behavior in coupled oscillators, death phenomenon should be avoided. On the basis of the odd number property, we never fail to avoid death if all the fixed points of individual oscillators satisfy the property. This is because the previous paper [6] proves that, for any coupling strength and delay time, the death never occurs at the fixed points which satisfy the property. It is obvious, however, that there are many oscillators whose fixed points do not satisfy the property. Hence, in order to avoid the death, we should change the characteristic of the fixed point so as to satisfy the property.

The main purpose of this paper is to provide a systematic procedure how to design a feedback controller which never induces amplitude death. The controller, which is added to

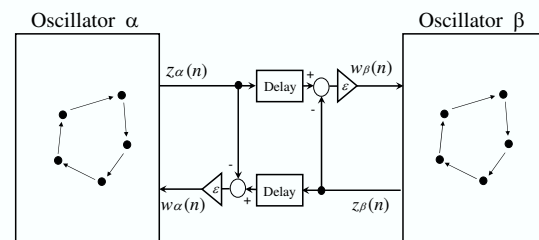


Figure 1: Oscillators coupled by time-delay connection

individual oscillators, is designed on the basis of the odd number property derived in [6]. The proposed controller has the following five features. (a) It is valid for high-dimensional oscillators. (b) It is the one-dimensional simple controller. (c) It can be designed by a systematic procedure. (d) It never induces death for any coupling strength and delay time. (e) It is valid for periodic, quasi-periodic, chaotic oscillators. Furthermore, this paper shows that the proposed controller works well on numerical simulations.

2. Oscillators coupled by time-delay connection

Consider two discrete-time oscillators,

$$\begin{cases} \mathbf{x}_{\alpha, \beta}(n+1) &= \mathbf{f}(\mathbf{x}_{\alpha, \beta}(n)) + \mathbf{b}_w w_{\alpha, \beta}(n) \\ z_{\alpha, \beta}(n) &= \mathbf{c}_z \mathbf{x}_{\alpha, \beta}(n) \end{cases}, \quad (1)$$

where $\mathbf{x}_{\alpha, \beta}(n) \in \mathbf{R}^m$ are the system variable, $w_{\alpha, \beta}(n)$, $z_{\alpha, \beta}(n) \in \mathbf{R}$ are the coupling signals, and $\mathbf{b}_w \in \mathbf{R}^m$, $\mathbf{c}_z \in \mathbf{R}^{1 \times m}$ are the coupling matrices. $\mathbf{f} : \mathbf{R}^m \rightarrow \mathbf{R}^m$ is the nonlinear map which has the hyperbolic fixed point $\mathbf{x}_f : \mathbf{f}(\mathbf{x}_f) = \mathbf{x}_f$. These oscillators are coupled by the diffusive connection,

$$\begin{cases} w_{\alpha}(n) &= \varepsilon \{ z_{\beta}(n - \tau) - z_{\alpha}(n) \} \\ w_{\beta}(n) &= \varepsilon \{ z_{\alpha}(n - \tau) - z_{\beta}(n) \} \end{cases}, \quad (2)$$

where $\varepsilon \in \mathbf{R}$ is the coupling strength and τ is the delay time. The fixed point of coupled system (1) with (2) is described by

$$\begin{bmatrix} \mathbf{x}_{\alpha}(n) & \mathbf{x}_{\beta}(n) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_f & \mathbf{x}_f \end{bmatrix}. \quad (3)$$

The fixed point location of the individual oscillators does not change even if they are coupled by (2). Amplitude death can be considered as the phenomenon where the coupled oscillators stop their oscillations and the coupling signals $w_{\alpha,\beta}$ become zero. Consequently, the stabilization of fixed point (3) is a necessary condition for death. The previous paper [6] derived a sufficient condition (odd number property) under which fixed point (3) is not stabilized for any coupling strength and delay time

Lemma 1 ([6]) Consider the Jacobi matrix of map f evaluated at hyperbolic fixed point \mathbf{x}_f ,

$$A = \left. \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}_f}.$$

If A has an odd number of real eigenvalues greater than 1, then fixed point (3) is not stabilized for any $\mathbf{b}_w, \mathbf{c}_z, \tau, \varepsilon$. Accordingly, amplitude death never occurs.

This Lemma was extended to continuous-time oscillators [7]. In order to see amplitude death on numerical simulations, we shall show the following two examples.

[Example 1] The two-dimensional nonlinear map [14]

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \gamma_1 x_1(1 - x_1 - x_2) & \gamma_2 x_2(1 + \gamma_3 x_1) \end{bmatrix}^T, \quad (4)$$

is used for the first numerical example. The parameters are set to $\gamma_1 = 2.50, \gamma_2 = 0.55, \gamma_3 = 5.00$. We obtain fixed point $\mathbf{x}_f = [0.1636 \ 0.4364]^T$. The eigenvalues of A are $\lambda_{1,2}(A) = 0.7955 \pm i0.6701$, then A does not satisfy Lemma 1. Therefore, from Lemma 1, we cannot guarantee whether amplitude death occurs or not. The coupling matrices and the delay time are set to $\mathbf{b}_w = [1 \ 0]^T, \mathbf{c}_z = [1 \ 0], \tau = 1$. Figures 2 (a)(b) show the bifurcation diagram and coupling signal w_α for the coupling strength ε . Death occurs for $\varepsilon \in [0.20, 0.74]$ in which the oscillations cease and the coupling signal w_α becomes zero.

[Example 2] Consider the three-dimensional map [15]

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \gamma_1 - x_2^2 - \gamma_2 x_3 & x_1 & x_2 \end{bmatrix}^T, \quad (5)$$

where the parameters are fixed at $\gamma_1 = 1.0, \gamma_2 = 0.1$. The map f has the two fixed points: $\mathbf{x}_f^{(1)} = 0.5913 [1 \ 1 \ 1]^T, \mathbf{x}_f^{(2)} = -1.6913 [1 \ 1 \ 1]^T$. The Jacobi matrices at the points are

$$A^{(1)} = \begin{bmatrix} 0 & -1.1825 & -0.1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 & 3.3825 & -0.1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

respectively. The eigenvalues of the matrices are $\lambda_{1,2,3}(A^{(1)}) = 0.0420 \pm i1.0899, -0.0841, \lambda_{1,2,3}(A^{(2)}) = -1.8538, 0.0296, 1.8242$. Since $A^{(2)}$ satisfies Lemma 1,

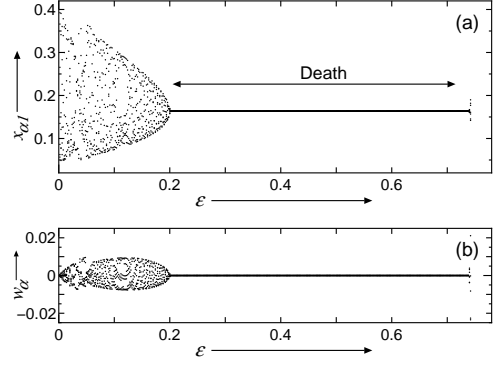


Figure 2: Bifurcation diagrams of coupled two-dimensional maps (4) for coupling strength ε . (a) System variable $x_{\alpha 1}(n)$. (b) Coupling signal $w_{\alpha}(n)$.

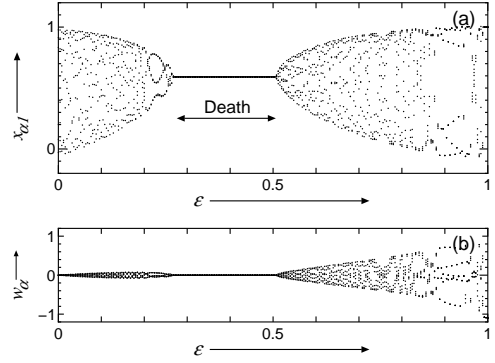


Figure 3: Bifurcation diagrams of coupled three-dimensional maps (5) for coupling strength ε . (a) System variable $x_{\alpha 1}(n)$. (b) Coupling signal $w_{\alpha}(n)$.

amplitude death never occurs at $\mathbf{x}_f^{(2)}$ for any coupling parameters $\mathbf{b}_w, \mathbf{c}_z, \varepsilon, \tau$. On the other hand, as $A^{(1)}$ does not satisfy Lemma 1, we cannot say whether death occurs or not. The coupling matrices and the delay time are set to $\mathbf{b}_w = [1 \ 0 \ 0]^T, \mathbf{c}_w = [1 \ 0 \ 0], \tau = 1$. The bifurcation diagram and the coupling signal for the coupling strength ε are shown in Figs. 3 (a)(b) respectively. We see that the death occurs for $\varepsilon \in [0.28, 0.49]$ in which the oscillations cease and the coupling signal w_α becomes zero.

3. Feedback control to avoid death

3.1. Control system

The input signals $u_{\alpha,\beta}(n)$ and output signals $y_{\alpha,\beta}(n)$ for control are added to oscillators (1) as follows:

$$\begin{cases} \mathbf{x}_{\alpha,\beta}(n+1) &= \mathbf{f}(\mathbf{x}_{\alpha,\beta}(n)) + \mathbf{b}_w w_{\alpha,\beta}(n) + \mathbf{b}_u u_{\alpha,\beta}(n) \\ z_{\alpha,\beta}(n) &= \mathbf{c}_z \mathbf{x}_{\alpha,\beta}(n) \\ y_{\alpha,\beta}(n) &= \mathbf{c}_y \mathbf{x}_{\alpha,\beta}(n) \end{cases}, \quad (6)$$

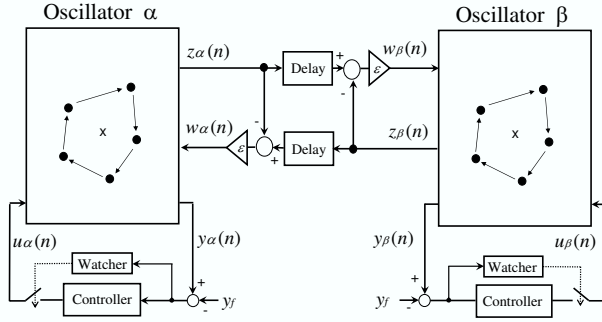


Figure 4: Coupled oscillators with feedback controllers

where $\mathbf{b}_u(n) \in \mathbf{R}^m$, $\mathbf{c}_y \in \mathbf{R}^{1 \times m}$ are the control matrices. The two controllers

$$\hat{u}_{\alpha, \beta}(n) = k(y_{\alpha, \beta}(n) - y_f), \quad (7)$$

are used for oscillators α and β respectively, where $y_f := \mathbf{c}_y \mathbf{x}_f$ and the controller gain $k \in \mathbf{R}$. Figure 4 illustrates the controlled oscillators.

In order that controller (7) works when $\mathbf{x}_{\alpha, \beta}(n)$ approaches the fixed point \mathbf{x}_f , we employ the watcher which operates as

$$u_{\alpha, \beta}(n) = \begin{cases} \hat{u}_{\alpha, \beta}(n) & |y_{\alpha, \beta}(n) - y_f| < \mu, \\ 0 & \text{otherwise.} \end{cases}$$

The criterion μ is set to a small positive value. It should be noted that this watcher might operate even when $\mathbf{x}_{\alpha, \beta}(n)$ are not in the neighborhood of \mathbf{x}_f . This is because the watcher uses only the output signals $y_{\alpha, \beta} := \mathbf{c}_y \mathbf{x}_{\alpha, \beta}(n)$, where \mathbf{c}_y might have zero elements.

3.2. Design of controller

Now we consider how to design the controller gain k such that death never occurs for any coupling parameters $\mathbf{b}_w, \mathbf{c}_z, \varepsilon, \tau$. This subsection provides a systematic procedure for designing the gain k .

We linearize oscillators (6) at fixed point (3):

$$\begin{cases} \xi_{\alpha, \beta}(n+1) &= \mathbf{A} \xi_{\alpha, \beta}(n) + \mathbf{b}_w w_{\alpha, \beta}(n) + \mathbf{b}_u u_{\alpha, \beta}(n) \\ \Delta z_{\alpha, \beta}(n) &= \mathbf{c}_z \xi_{\alpha, \beta}(n) \\ \Delta y_{\alpha, \beta}(n) &= \mathbf{c}_y \xi_{\alpha, \beta}(n) \end{cases},$$

where $\xi_{\alpha, \beta}(n) := \mathbf{x}_{\alpha, \beta}(n) - \mathbf{x}_f$, $\Delta z_{\alpha, \beta}(n) := z_{\alpha, \beta}(n) - \mathbf{c}_z \mathbf{x}_f$, $\Delta y_{\alpha, \beta}(n) := y_{\alpha, \beta}(n) - \mathbf{c}_y \mathbf{x}_f$. The dynamics of oscillators (6) controlled by (7) at fixed point (3) is governed by

$$\xi_{\alpha, \beta}(n+1) = (\mathbf{A} + \mathbf{b}_u k \mathbf{c}_y) \xi_{\alpha, \beta}(n) + \mathbf{b}_w w_{\alpha, \beta}(n).$$

We notice that oscillators (6) controlled by (7) never induce amplitude death if the Jacobi matrix,

$$\mathbf{A}' := \mathbf{A} + \mathbf{b}_u k \mathbf{c}_y, \quad (8)$$

satisfies Lemma 1. Consequently, if the gain k is chosen such that \mathbf{A}' satisfies Lemma 1, amplitude death never occurs for any coupling parameters $\mathbf{b}_w, \mathbf{c}_z, \varepsilon, \tau$.

Theorem 1 *If the controller gain k is chosen such that*

$$1 - k \mathbf{c}_y (\mathbf{I}_m - \mathbf{A})^{-1} \mathbf{b}_u < 0, \quad (9)$$

then hyperbolic fixed point (3) is not stabilized for any coupling parameters $\mathbf{b}_w, \mathbf{c}_z, \varepsilon, \tau$. Accordingly, death never occurs.

(Proof) We notice that if \mathbf{A}' satisfies Lemma 1, then the death never occurs. This proof shall show that \mathbf{A}' satisfies Lemma 1 if inequality (9) is held. Let us consider the characteristic function of (8), $g(\lambda) = \det[\lambda \mathbf{I}_m - \mathbf{A}']$. The roots $\lambda_{1, \dots, m}$ of the characteristic equation $g(\lambda) = 0$ are equivalent to the eigenvalues of the Jacobi matrix \mathbf{A}' . The function $g(\lambda)$ at $\lambda = 1$ is written as

$$g(1) = \det[\mathbf{I}_m - \mathbf{A}'] = \prod_{j=1}^m (1 - \lambda_j).$$

\mathbf{A}' satisfies Lemma 1 if and only if $g(1) < 0$. The function $g(1)$ can be rewritten as

$$\begin{aligned} g(1) &= \det[\mathbf{I}_m - \mathbf{A} - \mathbf{b}_u k \mathbf{c}_y] \\ &= \det[1 - k \mathbf{c}_y (\mathbf{I}_m - \mathbf{A})^{-1} \mathbf{b}_u] \\ &= 1 - k \mathbf{c}_y (\mathbf{I}_m - \mathbf{A})^{-1} \mathbf{b}_u. \end{aligned}$$

Consequently, when the gain k satisfies condition (9) (i.e., $g(1) < 0$), \mathbf{A}' satisfies Lemma 1. ■

3.3. Numerical examples

We shall design controller (7) by Theorem 1 for the two numerical examples shown in the previous section.

[Example 1] The nonlinear map \mathbf{f} and the coupling parameters $(\mathbf{b}_w, \mathbf{c}_z, \tau)$ are the same as the previous Example 1. The control parameters are set to $\mathbf{b}_u = [0 \ 1]^T$, $\mathbf{c}_y = [0 \ 1]$, $\mu = 0.02$. From these parameters, $\mathbf{c}_y (\mathbf{I}_m - \mathbf{A})^{-1} \mathbf{b}_u = 0.8333$ is estimated, then the gain k should be chosen such that $1 - 0.8333k < 0$. In this example, we use the gain $k = 2$. For this gain, the eigenvalues are $\lambda_{1,2}(\mathbf{A}') = 2.7753, 0.8156$: \mathbf{A}' satisfies Lemma 1. Figures 5 (a)(b) show the bifurcation diagram and the control signal $u_{\alpha}(n)$ respectively. It can be seen that the death does not occur in the region $\varepsilon \in [0.20, 0.74]$ where death occurs in Fig. 2 (a). This fact guarantees that the proposed controllers work well on numerical simulations.

[Example 2] The map \mathbf{f} and the parameters $(\mathbf{b}_w, \mathbf{c}_z, \tau)$ are the same as the previous Example 2. The control parameters are set to $\mathbf{b}_u = [0 \ 1 \ 0]^T$, $\mathbf{c}_y = [1 \ 0 \ 0]$, $\mu = 0.1$.

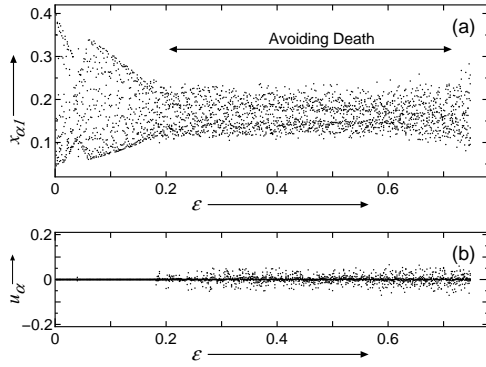


Figure 5: Bifurcation diagrams of coupled maps (4) with control for coupling strength ε . (a) System variable $x_{\alpha 1}(n)$. (b) Control signal $u_{\alpha}(n)$.

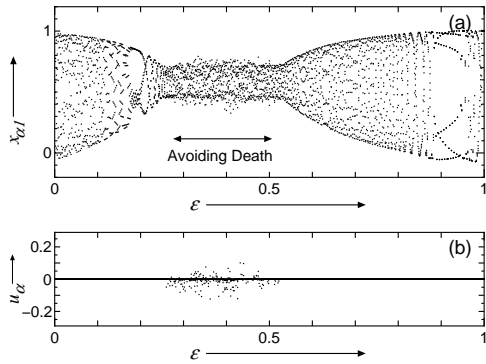


Figure 6: Bifurcation diagrams of coupled maps (5) with control for coupling strength ε . (a) System variable $x_{\alpha 1}(n)$. (b) Control signal $w_{\alpha}(n)$.

From $1 + 0.5619k < 0$, we choose the gain $k = -2$ satisfying Theorem 1. The matrix A' at $x_f^{(1)}$ satisfies Lemma 1 since $\lambda_{1,2,3}(A') = 1.1275, -1.0424, -0.0851$. Figures 6 (a)(b) show the bifurcation diagram and the control signal $u_{\alpha}(n)$ respectively. It can be seen that death does not occur in the region $\varepsilon \in [0.28, 0.49]$ where death occurs in Fig. 3 (a).

4. Conclusion

This paper provides a systematic procedure how to design a feedback controller which never induces amplitude death. The proposed controller is simple, but valid for high-dimensional oscillators. We show that the proposed controller works well on numerical simulations.

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