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A Deterministic Ising Model Evolving in Quantized Heat Bath

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Abstract—In this paper we present a dynamical Ising model evolving according to deterministic rules. We build the model by removing kinetic terms from local Hamiltonians of the Creutz model and putting it in a quantized heat bath. The bath is filled with heat particles which possess $\{+1, -1\}$-values and move pseudo-randomly like the Brownian particles. Evolution of spins is determined by comparing the interaction energy between the spins and the values the heat particles have. We found by numerical experiments that the steady states of the proposed model agree with the states of macroscopic statistical-mechanical models.

1. Introduction

A similarity between the Beysian formula and probability distribution in terms of steady states of Ising models makes it possible to apply the models to information processing such as image restoration and communication error correction [1]. In image restoration, we compute expectation of each spin's orientation in a steady state at which macroscopic Hamiltonian of an Ising model is minimized. Then, we can make a Beysian estimate of pixels of an original image before being contaminated by noise. The mean-field theory provides an iterative procedure to obtain the expectations. The procedure is described by a large-dimensional real difference equation. Thus, fast computation to attain the expectations requires a parallel processor for the iteration procedure. The hardware cost for the processor will be enormously high.

In 1986, Creutz proposed a Ising model which evolves dynamically according to a local deterministic rule [2]. Microscopic local Hamiltonian defined for each spin consists of its kinetic energy and the interaction energy between it and adjacent spins. Since the two kinds of energy are discretized, the deterministic evolution is described by simple equations with small integer variables. Then, hardware implementation of the Creutz model can be possible. As we mentioned above, the kinetic energy which represents heat energy is contained in local Hamiltonians. Thus, heat can flow only by the exchange between the kinetic and the interaction energy. The form of the local Hamiltonian and the flow process of heat make it difficult to control temperature independently and finely.

To overcome the difficulty we consider in this paper the following Ising model. We remove the kinetic terms from the local Hamiltonians of the Creutz model and put the model in a heat bath. The heat bath is filled with particles which possess $\{+1, -1\}$-values and move pseudo-randomly like the Brownian particles. The state of each spin evolves depending on the interaction energy and an average of the values of several particles near the spin.

We compare the steady states of this model with those of macroscopic statistical-mechanical models.

2. Structure and Evolution of the Ising Model

2.1. Structure

Figure 1 shows structure of the proposed model. It consists of three two-dimensional layers. In magnetic field layer, we set an inhomogeneous magnetic field. A magnetic field element $b_{i,j}\in\{+1, -1\}$ affects a spin at location (i, j). All the spins locate in spin layer. We denote their orientation by $\sigma_{i,j}(n)\in\{+1, -1\}$, n:time index. Each of the spins has a local Hamiltonian $H_{i,j}$ which depends on $b_{i,j}$, $\sigma_{i,j}(n)$ and $\sigma_{i\pm1,j\pm1}(n)$. Location (i, j) in heat bath layer is always occupied with four heat particles. We denote values the particles at (i, j) possess by $t^k_{i,j}(n)\in\{+1, -1\}$, k=1, ..., 4. The particles move pseudo-randomly in heat bath layer like the Brownian particles. Thus, value $t^k_{i,j}(n)$ also changes pseudo-randomly.

2.2. Heat bath and heat particles

We will explain more about the heat bath and the heat particles [3]. Figure 2 illustrates how we construct the heat bath. We use cells in Fig. 2(a) which have two inputs $a_l(n)$,$a_r(n)\in\{+1, -1\}$, two outputs $u_l(n)$,$u_r(n)\in\{+1, -1\}$ and one internal state $q(n)\in\{+1, -1\}$. The cells operate according to Tab. 1. Since $a_l(n)+a_r(n)=u_l(n)+u_r(n)$, we can express the cell operation by the following simile. Two virtual particles carrying "±1" enter the cell at time $n-1$, pass through the cell in parallel ($a_l \to u_l$, $a_r \to u_r$) or cross each other in the cell ($a_l \to u_r$, $a_r \to u_l$), and go out of the cell at time n. The parallel passing or crossing is determined by the following Boolean

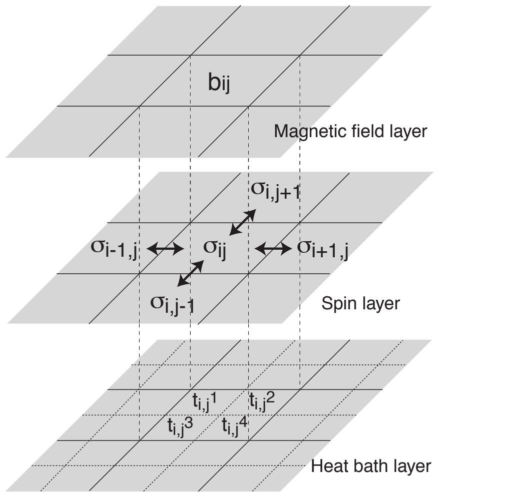

Figure 1: The structure of the dynamical Ising model.

expression:

$$parallel/\overline{cross} = \overline{a_l(n-1) \oplus q(n-1)} \qquad (1)$$

In this equation $a_l(n)$ and $q(n)$ take "logical 1/0" if they are "+1/ − 1". State $q(n)$ is considered to take "+1" or "−1" at a probability of 1/2 since the state is reversed every time when $a_l(n)+a_r(n)=0$. Then, probability of parallel passing or crossing is estimated to be 1/2.

One-dimensional (1-D) arrays are built as shown in Fig. 2(b). A 2-D array is built by arranging the 1-D arrays in lattice and connecting them by pairs of cells. Outputs from every two cells in each 1-D array are supplied to their adjacent four cells through the pair of cells as shown in Fig. 2(c). We denote the cell which belongs to j-th row or i-th column array and locates just above or below a pair of cells by $cell^r_{i,j}$ or $cell^c_{i,j}$. A heat particle at $cell^r_{i,j}$ or $cell^c_{i,j}$ moves for 3 time steps to one of four cells $cell^r_{i\pm1,j}$, $cell^c_{i,j\pm1}$ at a probability of 1/8 or goes back to $cell^r_{i,j}$ or $cell^c_{i,j}$ at a probability of 1/4. Then, the particles move pseudo-randomly in heat bath layer like the Brownian particles with average zero and variance $n/12$ in terms of displacement. Let $t^1_{i,j}, \cdots, t^4_{i,j}$ be the values of four particles in $cell^r_{i,j}$,$cell^c_{i,j}$. Because systems of many Brownian particles are diffusion systems, local temperature defined by

$$Tl_{i,j} = \frac{1}{4}\frac{1}{(2loc+1)^2}\sum_{l=-loc+i}^{loc+i}\sum_{m=-loc+j}^{loc+j}\sum_{k=1}^{4} t^k_{l,m} \qquad (2)$$

is governed by a two-dimensional linear parabolic partial differential equation, what is called a diffusion equation.

According to statistical mechanics, averaged kinetic energy $< H_{kinetic} >$ of particles in a system is given

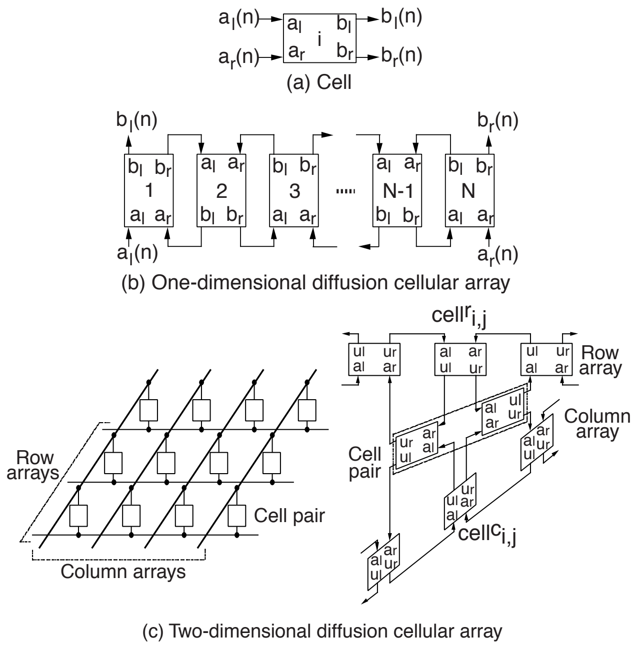

Figure 2: The heat bath.

Table 1: The operation rule of cells in heat bath.

a_l(n-1)	a_r(n-1)	b_l(n)	b_r(n)	q(n)
-1	-1	-1	-1	q(n-1)
-1	1	q(n-1)	-q(n-1)	-q(n-1)
1	-1	q(n-1)	-q(n-1)	-q(n-1)
1	1	1	1	q(n-1)

at temperature T by

$$< H_{kinetic} >= \frac{\int_0^\infty He^{-H/k_BT}dH}{\int_0^\infty e^{-H/k_BT}dH} = k_BT \qquad (3)$$

It should be noted that the heat layer does not satisfy Eq.(3) because the heat particles possess discrete values $\{+1, -1\}$.

2.3. Updating spins

We show here how the spins evolve. We define local Hamiltonian for each spin by

$$\begin{aligned} H_{i,j}(n) = &-Bb_{i,j}\sigma_{i,j}(n) - J\sigma_{i,j}(n)\{\sigma_{i+1,j}(n) \\ &+ \sigma_{i-1,j}(n) + \sigma_{i,j+1}(n) + \sigma_{i,j-1}(n)\} \end{aligned} \qquad (4)$$

where B is an integer and we set J=1. Suppose that the spin at (i,j) is reversed. Then, the local Hamiltonian decreases or increases by $\Delta H_{i,j}=-2H_{i,j}(n)$. We determine whether we actually reverse the spin or not by comparing $\Delta H_{i,j}$ and $T_{i,j}=(1/4)\sum_{k=1}^{4} t^k_{i,j}$. Table 2 shows the reverse rule. We can change the critical temperature at which the Ising model causes phase transition by shifting the boundary between R(reverse) and H(hold).

Table 2: Spin evolution rule.

$4T_{ij}$ \ ΔH_{ij}	8	4	0	−4	−8
4	R	R	R	R	R
2	H	R	R	R	R
0	H	H	R	R	R
−2	H	H	H	R	R
−4	H	H	H	H	R

R: Reverse, H:Hold

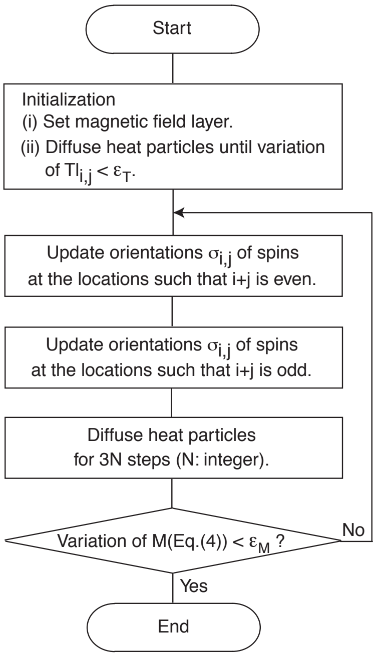

Figure 3: The iteration procedure.

2.4. Evolution of the model

Figure 3 illustrates the Ising model's execution sequence. In the initialization, we set the magnetic field layer and make the heat bath in steady state. A half of the spins at locations such that $i + j$ is even and other half of the spins at locations such that $i + j$ is odd evolve alternately according to Tab. 2. After the update of spins the heat bath iterates diffusion operation for $3N$ times (N: integer). The diffusion speed relative to the spin evolution can be higher as we set N larger.

3. Numerical Experiments

In experiments 1 and 2, we observe steady states of spins at zero magnetic field. In experiment 3, we investigate steady behavior of spins under inhomogeneous magnetic field.

Experiment 1: Spins in closed heat bath
The size of the three layers is 64×64, that is, location indices are $1 \leq i, j \leq 64$. At all the ends of row and column arrays of the heat bath inputs and outputs of the cells are connected such that $a_l(n) = u_l(n-1)$ or $a_r(n) = u_r(n-1)$, which corresponds to a Neumann condition $\partial Tl/\partial x = 0$. After the heat bath has uniform temperature distribution $Tl_{i,j} \simeq T$(: constant), the spins start evolution and get into a steady state. Then, magnetization M given by

$$M = \frac{1}{L^2} \sum_{i,j=1}^{L} \sigma_{i,j}(n), \quad L = 64 \tag{5}$$

becomes almost constant. Figures 4 shows the spin orientations at T=−0.4, −0.1, 0, +0.1, and +0.4. The spin clusters become small at higher temperatures, which qualitatively agrees with the results of Monte Carle analysis of an Ising model whose Hamiltonian is defined macroscopically. Figure 5 shows a magnetization M versus temperature T curve. According to statistical-mechanical analysis, ideal $M - T$ curve should be $M \propto T^{1/8}$. We see that the curve in Fig. 5 satisfies the ideal $M - T$ relation only in a small region, $-0.2 \leq T \leq 0.0$. This is because the heat layer does not satisfy Eq. (3). In this experiment we sometimes failed to make magnetization M converge on a point of the curve.

Experiment 2: Spins in open heat bath
The size of the three layers is 40×200. Inputs of cells at the left and right ends of row arrays of the heat bath are $a_l(n) = -1$ and $a_r(n) = +1$, which corresponds to a Dirichlet condition $Tl = \pm 1$. After the local temperature $Tl_{i,j}$ is distributed with a gradient of $(1-(-1))/200$ in row direction, we have the spin evolve. The spins became steady as shown in Fig. 6. We see that the spins form smaller clusters in higher temperature region.

Experiment 3: Spins under inhomogeneous magnetic field
We investigate whether the model has a capability to restore image when a degraded image is set in the magnetic field layer. The size of every layer is 64×64. The boundary of the heat bath has the same condition with that of the model in Experiment 1. The parameter B of the local Hamiltonians is fixed to be 2. This elementary experiment does not include annealing process. Although we have to decide carefully the parameters and the temperature in order to give the model high capability of restoring degraded images, we do not discuss here how to decide the parameters and how to anneal spins.

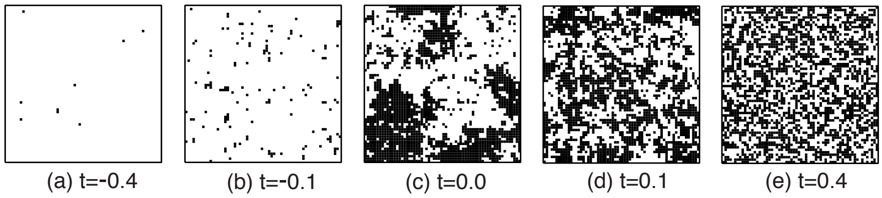

Figure 4: The steady states of spins in closed heat bath.

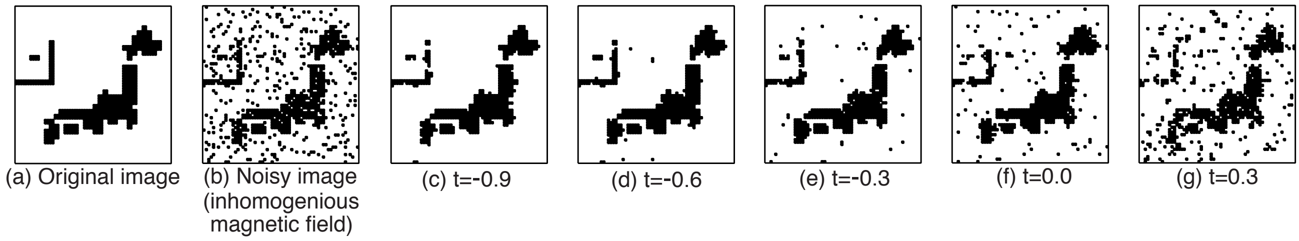

Figure 7: The steady states of spins when an inhomogeneous magnetic field is applied.

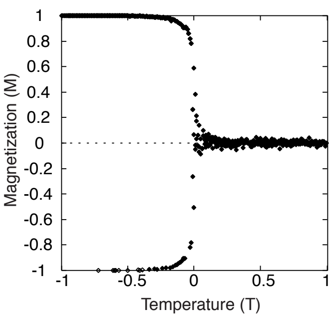

Figure 5: Magnetization versus temperature.

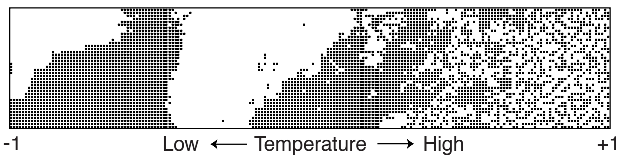

Figure 6: The steady spin layer in open heat bath.

Figure 7 shows original monochrome image (a), degraded image (b), and steady states of the spins (c)-(g) at T=-0.9, -0.6, -0.3, 0.0, and 0.3. Degraded image (b) contains spatially independent noise. In 64×64 pixels, 9.2% of pixels are different between Figs. 7(a) and (b). The error pixel rates in Figs. 7(c),(d) are decreased to 1.22 and 1.34%. We may consider these figures as restored images. They are obtained after 30000 iterations. Thus, we may restore images in 3msec. when we develop a hardware iterating the procedure shown in Fig. 3 at a clock rate of 10MHz.

4. Conclusions

We have built a dynamical and deterministic Ising model by removing kinetic terms from the local Hamiltonians of the Creutz model and putting the model in a quantized heat bath. We found in Sect. 3 that the steady behavior of the proposed model agrees with the states of macroscopic statistical-mechanical models. To analyze macroscopic fluctuation of the steady states and transient behavior before getting into the steady states is our future work. We also found that the proposed model has a primitive capability of restoring degraded images. To determine optimal parameters of local Hamiltonians and to control temperature optimally are also our future works toward parallel image restoration machines.

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