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Self-Similarity in Explosive Transition to/from Synchronization in Random Networks

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Abstract—In the present report we consider the remarkable phenomenon of the explosive synchronization in complex networks of oscillators. We have shown that this phenomenon is a consequence of the self-similarity in the stability loss of the synchronous clusters of different size. The manifestation of the self-similarity can be revealed through the processes of the network synchronous state destruction. As a sample system the random network of Kuramoto oscillators has been considered. We have shown that the destruction of the synchronous state of the random network goes step by step through the self-similar configurations of interacting oscillators.

1. Introduction

Coupled networks of phase oscillators offer a benchmark description in a large number of natural systems, such as neurons in human brain, cardiac pacemaker cells, power grids, etc [1]. The synchronization phenomenon plays a key role for the collective dynamics of the node elements of networks, whereas the transition between the asynchronous and synchronous states is of fundamental importance for understanding the core mechanisms of the behavior of the interacting oscillators [2, 3, 4] and complex networks [5, 6, 7, 8, 9].

From the thermodynamical point of view the passage from the asynchronous state to the synchronous oscillations may be considered as the phase transition. There are two types of the phase transition are distinguished: the abrupt transitions to synchronized states (called as the first-order transition) and the continuous phase transition (the second-order phase transition) [10]. Typically, in the complex networks the smooth phase transition takes place when the coupling strength between nodes grows and the asynchronous oscillatory motion becomes synchronized [8, 5, 11]. At the same time, the discontinuous transformation (so called the explosive synchronization, when the network does not pass through the intermediate partial synchronization but rather jumps from the asynchronous to synchronized motion and vice versa) being the first-order phase transition can also be observed for the complex networks [12, 13, 14].

2. Explosive transition

The occurrence of the explosive synchronization in the networks of oscillators is known to be observed for different network architectures. The first-order phase transition to synchronous dynamics takes place in networks with all-to-all coupled node elements [15], in random networks [16], in the networks with scale-free topology of links [13, 17, 18] (including scale-free networks with the time-delayed coupling [19]). The explosive synchronization is also observed in the networks of the adaptively coupled oscillators [20]. Although the most popular models for the explosive synchronization study are the networks of Kuramoto oscillators [21, 22], the spontaneous explosive emergent behavior has also been observed for the other types of oscillators placed in the nodes of complex networs, e.g., for the generalized Kuramoto oscillators [23] and piecewise Rössler units [12]. In other words, the discontinuous explosive transition can be considered as a generic feature of phase oscillator networks.

For the first-order transition to be realized, the certain conditions (being distinct for different network topologies) must be fulfilled. In this case both the establishment and destruction of the synchronous behavior of the network elements are characterized by the abrupt transformation of network dynamics, with hysteresis (in several cases) being observed. In the present report we have shown that the phenomenon of the abrupt transition in networks is a consequence of the self-similarity in the stability loss of the synchronous clusters of different size. As a sample system to be considered the random network of Kuramoto oscillators [21, 22] has been chosen.

3. Results

Our results have been obtained with a random network of Kuramoto oscillators which behave according to the following dynamic equation

$$\dot{\varphi}_i = \omega_i + \frac{\lambda}{N}\sum_{j=1}^{N} a_{ij}\sin\left(\varphi_j - \varphi_i\right), \qquad (1)$$

where N is the number of coupled oscillators in network, φ_i and ω_i are the phase and natural frequency of i-th oscillator, respectively, λ is the coupling strength, and $\{a_{ij}\}$ are the elements of the adjacency matrix A that uniquely defines the nodes' interactions ($a_{ij} = a_{ji} = 1$ if oscillators i and j are connected with each other and zero otherwise). The natural frequencies ω_i are supposed to be different and, therefore, the synchronized motion appears only above some coupling strength threshold λ_c. We consider the case of evenly spaced natural frequencies

$$\omega_i = -\Omega + \frac{\Omega}{N}(2i - 1), \qquad (2)$$

where $\Omega = 0.5$, $i = 1, \ldots, N$. In other words, the frequency distribution $g(\omega)$ should be considered as symmetric and centered at zero

$$g(\omega) = \begin{cases} \dfrac{1}{2\Omega} & \text{for } |\omega| \leq \Omega \\ 0 & \text{for } |\omega| > \Omega. \end{cases} \qquad (3)$$

The adjacency matrix A characterizing the topology of network links represents Erdös and Rényi (ER) random graph [24] obtained by the well-known algorithm which consists in connecting each couple of nodes with a probability $0 < p < 1$ [1]. In our work we have used random networks consisting of $N = 5 \times 10^2, 10^3$ and 5×10^3 elements with the probabilities $p = 0.1, 0.3, 0.5, 0.7, 0.9$.

We have examined the processes of the synchronous motion destruction in the random networks both theoretically and numerically. We have shown that for the sufficiently large random networks (more precisely, in the limit of the infinite population, $N \to \infty$) the synchronous state of random network loses its stability at

$$\lambda_c = \frac{4N\Omega}{\pi\langle k\rangle}, \qquad (4)$$

where $\langle k\rangle = Np$. In the limit of $p \to 1$ the threshold value of the coupling strength (4) coincides with the critical value obtained for all-to-all connected network [15] as well as with the value where the incoherent solution becomes unstable according to the classical result [25] for all unimodal distributions, $\lambda_c = 2/\pi g(0)$. Above the critical coupling strength value, λ_c, all network oscillators are synchronized and the whole network can be considered as one synchronous cluster of size N. At the threshold coupling (or, more precisely, just below, $\lambda \to \lambda_c-$) the synchronous cluster start destroying. Having denoted the size of the synchronous cluster (i.e., the number of the network oscillators showing the synchronous behavior) as $\mathcal{N}$ and analysed the stability properties of the synchronous cluster of size $\mathcal{N} \leq N$ we have found that it also becomes unstable at λ_c. In other words, when the destruction process of synchronous state of the network begins, the part of oscillators start moving asynchronously and, as a consequence, the coherent structure of size N is replaced by the smaller coherent structure (consisting of $\mathcal{N}(t)$ synchronous oscillators, $\mathcal{N}(0) = N$) which is also unstable. Having examined the evolution of the coherent structure during the cluster destruction processes, we have found that the size of the synchronous cluster $\mathcal{N}(t)$ decreases linearly with the time growth. So, when the explosive transition from the synchronization to asynchronous dynamics takes place, the coherent cluster of synchronous oscillators passes sequentially through the different self-similar configurations of size $\mathcal{N}(t)$, with all of them becoming unstable at one and the same critical point λ_c.

Remarkably, that the probability distributions $p_{\mathcal{N}}(\varphi)$ of the instantaneous phases φ_i of oscillators consisting the synchronous cluster of size $\mathcal{N}$ also demonstrate the self-similarity properties. The profile of the phase distributions for synchronized nodes remains unchanged during the abrupt transition from the synchronous state to asynchronous motion of network oscillators at fixed value of the coupling strength $\lambda \to \lambda_c-$, i.e.

$$p_N(\varphi) = p_{\mathcal{N}(t)}(\varphi) = \text{const}, \quad \forall t. \qquad (5)$$

4. Conclusion

In conclusion, in the present paper we have shown that the remarkable phenomenon of the explosive synchronization in complex network of oscillators is connected tightly with the self-similarity property. We have shown that the abrupt transition from the synchronized state of the whole random network of oscillators is a consequence of the self-similarity in the stability loss of the synchronous clusters of different size. The manifestation of the self-similarity has been revealed through the examination of the processes of the synchronous cluster destruction. We have shown that the destruction of the synchronous state of the random networks goes step by step through the self-similar configurations of synchronous clusters of interacting oscillators. Although as a sample system the random network of Kuramoto oscillators has been chosen, we expect that the very same mechanism of the abrupt transition from/to synchronization should be observed in the networks of other oscillator types. We believe that the deep insight into the core mechanisms (including the self-similarity phenomenon) of the explosive transitions being the boundaries dividing the synchronous and asynchronous dynamics in the complex networks gives the substantial profit for both the theory of complex networks and the practical applications in the wide spectrum of the human activity.

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References

[1] S. Boccaletti, V. Latora, V. Moreno, M. Chavez, and D. U. Hwang. "Complex networks: Structure and dynamics". *Physics Reports*, vol. 424, pp. 175–308, 2006.

[2] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths. "From phase to lag synchronization in coupled chaotic oscillators". *Phys. Rev. Lett.*, vol. 78(22), 4193–4196, 1997.

[3] A. E. Hramov and A. A. Koronovskii. "Generalized synchronization: a modified system approach". *Phys. Rev. E*, vol. 71(6), 067201, 2005.

[4] A. E. Hramov, A. A. Koronovskii, and M. K. Kurovskaya. "Two types of phase synchronization destruction". *Phys. Rev. E*, vol. 75(3), 036205, 2007.

[5] A. Arenas, A. Diaz-Guilera, and C.J. Pérez-Vicente. "Synchronization reveals topological scales in complex networks". *Phys. Rev. Lett.*, vol. 96, 114102, 2006.

[6] C. Zhou and J. Kurths. "Hierarchical synchronization in complex networks with heterogeneous degrees". *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 16(1), 015104, 2006.

[7] J. Gomez-Gardenes, Y. Moreno, and A. Arenas. "Paths to synchronization on complex networks". *Physical Review Letters*, vol. 98, 034101, 2007.

[8] A. Arenas, A. Diaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou. "Synchronization in complex networks". *Physics Reports*, vol. 469(3), pp. 93–153, 2008.

[9] W. Liu, Y. Wu, J. Xiao, and M. Zhan. "Effects of frequency-degree correlation on synchronization transition in scale-free networks". *EPL (Europhysics Letters)*, vol. 101(3), 38002, 2013.

[10] S. Boccaletti, J.A. Almendral, S. Guan, I. Leyva, Z. Liu, I. Sendiña Nadal, Z. Wang, and Y. Zou. "Explosive transitions in complex networks structure and dynamics: Percolation and synchronization". *Physics Reports*, vol. 660, pp. 1–94, 2016.

[11] Y. Moreno and A. F. Pacheco. "Synchronization of Kuramoto oscillators in scale-free networks". *EPL (Europhysics Letters)*, vol. 68(4), 603, 2004.

[12] I. Leyva, R. Sevilla-Escoboza, J. M. Buldu, I. Sendina-Nadal, J. Gomez-Gardenes, A. Arenas, Y. Moreno, S. Gomez, R. Jaimes-Reategui, and S. Boccaletti. "Explosive first-order transition to synchrony in networked chaotic oscillators". *Phys. Rev. Lett.*, vol. 108, 168702, 2012.

[13] J. Gomez-Gardenes, S. Gomez, A. Arenas, and Y. Moreno. "Explosive synchronization transitions in scale-free networks". *Phys. Rev. Lett.*, vol. 106, 128701, 2011.

[14] I. Leyva, A. Navas, I. Sendina-Nadal, J. A. Almendral, J. M. Buldu, M. Zanin, D. Papo, and S. Boccaletti. "Explosive transitions to synchronization in networks of phase oscillators". *Scientific Reports*, vol. 3, 1281, 2013.

[15] D. Pazó. "Thermodynamic limit of the first-order phase transition in the Kuramoto model". *Phys. Rev. E*, vol. 72, 046211, 2005.

[16] L. Zhu, L. Tian, and D. Shi. "Criterion for the emergence of explosive synchronization transitions in networks of phase oscillators". *Phys. Rev. E*, vol. 88, 042921, 2013.

[17] T. K. D. Peron and F.A. Rodrigues. "Determination of the critical coupling of explosive synchronization transitions in scale-free networks by mean-field approximations". *Phys. Rev. E*, vol. 86, 056108, 2012.

[18] Y. Zou, T. Pereira, M. Small, Z. Liu, and J. Kurths. "Basin of attraction determines hysteresis in explosive synchronization". *Phys. Rev. Lett.*, vol. 112, 114102, 2014.

[19] T. K. D. Peron and F.A. Rodrigues. "Explosive synchronization enhanced by time-delayed coupling". *Phys. Rev. E*, vol. 86, 016102, 2012.

[20] M.M. Danziger, O. I. Moskalenko, S. A. Kurkin, X. Zhang, S. Havlin, and S. Boccaletti. "Explosive synchronization coexists with classical synchronization in the Kuramoto model". *Chaos*, vol. 26, 065307, 2016.

[21] Y. Kuramoto. "Self-entrainment of a population of coupled nonlinear oscillators". *Lect. Notes in Physics*, vol. 30, pp. 420–422, 1975.

[22] J.A. Acebrón, L. L. Bonilla, C.J. Pérez-Vicente, F. Ritort, and R. Spigler. "The Kuramoto model: A simple paradigm for synchronization phenomena". *Rev. Mod. Phys.*, vol. 77, pp. 137–185, 2005.

[23] X. Hu, S. Boccaletti, W. Huang, X. Zhang, Z. Liu, S. Guan, and C. H. Lai. "Exact solution for first-order synchronization transition in a generalized Kuramoto model". *Scientific Reports*, vol. 4, 7262, 2014.

[24] P. Erdös and A. Rényi. "On random graphs I". *Publ. Math. Debrecen*, vol. 6, pp. 290–297, 1959.

[25] S.H. Strogatz and R.E. Mirollo. "Stability of incoherence in a population of coupled oscillators". *Journal of Statistical Physics*, vol. 63(3), pp. 613–635, 1991.