

Structurally stable PWL approximation of a simple dynamical system

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Abstract—This paper concerns the piecewise-linear (PWL) approximation of the cusp normal form, which is a well-known nonlinear dynamical system. Even if applied to a simple example, the method can be easily generalized and allows one to define a strategy to calculate the optimal coefficients for the PWL approximation. It is possible to choose either an approximation accuracy almost uniform all over the approximation domain, or an approximation particularly accurate in the neighborhoods of significant sets of points (bifurcation curves or invariant sets), or a compromise between these two situations. The numerical results in the considered example confirm the reliability of the proposed approach.

1. Introduction

This paper deals with the piecewise-linear (PWL) approximation (in view of structurally stable circuit implementations) [1] of known nonlinear dynamical systems with bounded dynamics governed by the following set of ordinary differential equations:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t); \mathbf{p}) \quad (1)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ (state vector), $\mathbf{p}(t) \in \mathbb{R}^q$ (parameter vector), $\mathbf{f} : S \subset \mathbb{R}^{n+q} \rightarrow \mathbb{R}^n$ (vector field), S is a limited compact domain, and $\dot{\mathbf{x}}$ denotes the time derivative of $\mathbf{x}(t)$. All the vectors are intended as column vectors. In view of the circuit synthesis of such kind of dynamical systems, first of all we aim to approximate the vector field \mathbf{f} through a linear combination of basis functions having a direct circuit implementation. Then, generally speaking, for any component (say f) of the vector field \mathbf{f} , our reference model can be written as follows:

$$f_{PWL}(\mathbf{y}; N) = \sum_{k=1}^N w_k(N) \varphi_k(\mathbf{y}; N) \quad (2)$$

where $\mathbf{y} = (\mathbf{x}^T; \mathbf{p}^T)^T$ and N is the (integer) number of basis functions $\varphi_k(\mathbf{y}; N)$ whose sum (weighted through the coefficients $w_k(N)$) provides an approximation of the vector field component f (of course, each component of \mathbf{f} will have its own set of coefficients $w_k(N)$).

In particular, we shall refer to PWL models that have direct circuit implementations [2, 3] and are based on *a priori* simplicial partitions of the domain, *i.e.*, triangulations formed by rectangular partitions plus north-east diagonals [4, 5]. In this case, the number N of basis functions (which depends directly on the number of subdomains the domain S is partitioned in) can be fixed in a first step by some heuristic criteria, *e.g.*, simply on the basis of function inspection [4, 5]. The coefficients w_k are determined in a second step by minimizing a proper cost function [4, 5]. In the absence of *a priori* knowledge, the number of samples that are necessary to have an accurate approximation of f would grow exponentially with the number of dimensions. However, if the function to be approximated is known, it is possible to fix a reasonable number of subdivisions along each dimensional component of the domain. The main advantage of the simplicial approach is its direct circuit implementation [2, 3], which can be particularly useful whenever we aim to emulate the behaviors of dynamical systems made up of a large number of elementary units [1].

In this paper, we shall propose a new functional for the computation of the coefficients w_k . Such a functional (that owing to Eq. (2) and for a fixed N and vector $\varphi_k(\mathbf{y}; N)$ can be also viewed as a cost function for the coefficients w_k) is made up of two additive terms. A first term is the usual mean square approximation error evaluated over the domain S . The minimization of such a term provides PWL approximations whose accuracy is almost uniform over S . To this first component of the cost function, we shall add a penalty term to force the approximation to be particularly accurate in a neighborhood of a subset of S where the vector field is critical for the behaviors of the dynamical system (1). For instance, such a term can be a bifurcation curve in a system really depending on parameters or an invariant set in a system with fixed parameters. The balance between the two terms of the cost function is governed by a coefficient that will be tuned by minimizing a proper quality factor, that measures the approximation quality. Of course, both the penalty term and the quality factor can be chosen on the basis of the most significant sets of points of the

considered system. The example proposed in this paper is a very simple codimension-two normal form, *i.e.*, the cusp normal form. To verify the structural stability – in a given limited domain – of the original system to the perturbation induced by the approximation, we shall carry out a bifurcation analysis by resorting to some packages for numerical continuation [6,7]. As an essential prerequisite for using such methods is vector field smoothness, we shall replace *a posteriori* the PWL vector field with a piecewise-smooth (PWS) version of it [1].

2. PWL approximation

We shall denote by \mathbf{f}_{PWL} a continuous PWL approximation of the vector field \mathbf{f} over the $(n+q)$ -dimensional domain S , *i.e.*, $\mathbf{f}_{PWL} : S \rightarrow \mathbb{R}^n$, where

$$S = \{\mathbf{y} \in \mathbb{R}^{n+q} : a_i \leq y_i \leq b_i, i = 1, \dots, n+q\} \quad (3)$$

Each dimensional component y_i of the domain S can be either a state vector or a parameter vector component, and can be subdivided into m_i subintervals of amplitude $(b_i - a_i)/m_i$. By adding the north-east diagonals to the obtained configuration, each hyperrectangle is in turn partitioned into $n!$ non-overlapping simplices. This partition is called boundary configuration H [4] and depends on the vector $\mathbf{m} = [m_1, \dots, m_{n+q}]^T$. Consequently, S is partitioned (*simplicial partition* S_H) into $\prod_{i=1}^{n+q} m_i$ hyperrectangles and contains $N = \prod_{i=1}^{n+q} (m_i + 1)$ vertices. The domain associated with a boundary configuration H can be completely described by the triplets (a_i, b_i, m_i) , $i = 1, \dots, n+q$. The class of continuous PWL functions \mathbf{f}_{PWL} that are linear over each simplex constitutes an N -dimensional metric space $PWL[S_H]$, which is defined by the domain S , its simplicial partition, and a proper inner product (see [5] for details). According to Eq. (2), each function belonging to $PWL[S_H]$ can be represented as a sum of N basis functions (organized according to a given criterion into a vector), weighted by an N -length coefficient vector \mathbf{w} . For a fixed \mathbf{m} , the coefficients \mathbf{w} determine the shape of \mathbf{f}_{PWL} .

3. PWL approximation of the Cusp normal form

The cusp normal form is:

$$\dot{x} = f(x; \mathbf{p}) = p_1 + p_2 x - x^3 = p_1 + h(x; p_2) \quad (4)$$

The variables p_1 and p_2 are control parameters, whereas x is the state variable. Figure 1(a) shows the bifurcation diagram of the normal form (4), where the two regions A (corresponding to the presence of three equilibrium points) and B (with just one equilibrium point) are evidenced (see [7, 1] for details).

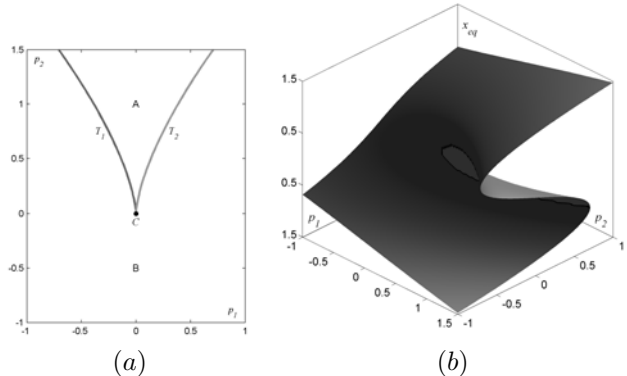


Figure 1: Cusp normal form (4). (a) – Bifurcation diagram. (b) – Equilibrium manifold $x_{eq}(p_1, p_2)$.

The two curves T_1 and T_2 are saddle-node bifurcation curves and henceforth we shall denote their union as γ . Such a diagram is the projection of the folds of the cusp equilibrium manifold (shown in Fig. 1(b)) on the parameter plane. Owing to the simple structure of Eq. (4), we shall focus on the PWL approximation $h_{PWL}(x; p_2)$ of the bivariate nonlinear scalar function $h(x; p_2)$ (*i.e.*, $n = 1$ and $q = 1$), thus considering the parameter p_1 as a simple offset. The considered domain is $S = \{\mathbf{y} [= (x; p_2)] \in \mathbb{R}^2 : a_i \leq y_i \leq b_i, i = 1, 2\}$, with $a_1 = a_2 = -1.5$ and $b_1 = b_2 = 1.5$. The rectangular partition is obtained by fixing *a priori* the vector \mathbf{m} . The coefficients w_k are determined in a second step by minimizing a proper cost function.

3.1. Cost function

The cost function considered in this paper is different from those used in previous works [5, 1] and can be written as follows:

$$F(\mathbf{w}; \lambda) = \int_S [h(x; p_2) - h_{PWL}(x; p_2)]^2 ds + \lambda \int_\gamma [h(x; p_2) - h_{PWL}(x; p_2)]^2 d\gamma \doteq F_1(\mathbf{w}) + \lambda F_2(\mathbf{w}) \quad (5)$$

where $F_1(\mathbf{w})$ is the cost function used in [5], that constrains the solution to approximate $h(x; p_2)$ uniformly well all over the domain S , whereas $F_2(\mathbf{w})$ forces the solution to remain close to h in the C^0 distance [7] around the curve γ . In other terms, we aim to constrain the PWL approximation to be reasonably accurate all over S and particularly accurate in a neighborhood of the bifurcation curve γ , with the aim of finding a structurally stable approximation even with relatively coarse simplicial partitions. The compromise between the two goals is ruled by the coefficient λ , that will be estimated through the minimization of the quality factor defined in Sec. 3.2.

For a given λ , the optimal weights vector \mathbf{w} can be obtained by imposing $\frac{\partial F}{\partial w_j} = 0$ for $j = 1, \dots, N$.

Making reference to Eq. (2), we obtain:

$$\begin{aligned}\frac{\partial F_1}{\partial w_j} &= -2 \int_S h \varphi_j dp_2 dx + 2 \sum_k w_k \int_S \varphi_k \varphi_j dp_2 dx \\ \frac{\partial F_2}{\partial w_j} &= -2 \int_\gamma h \varphi_j d\gamma + 2 \sum_k w_k \int_\gamma \varphi_k \varphi_j d\gamma\end{aligned}\quad (6)$$

Then, we can impose the optimality conditions $\frac{\partial F}{\partial w_j} = 0$ in the following form:

$$-B_j + \sum_k A_{jk} w_k + \lambda \left\{ -\tilde{B}_j + \sum_k \tilde{A}_{jk} w_k \right\} = 0 \quad (7)$$

where $A_{jk} = \int_S \varphi_k \varphi_j dp_2 dx$, $B_j = \int_S h \varphi_j dp_2 dx$, $\tilde{A}_{jk} = \int_\gamma \varphi_k \varphi_j d\gamma$, and $\tilde{B}_j = \int_\gamma h \varphi_j d\gamma$.

Equation (7) can be expressed in compact matrix form as $\hat{A}\mathbf{w} = \hat{B}$, where $\hat{B} = B + \lambda\tilde{B}$ and $\hat{A} = A + \lambda\tilde{A}$. Thus, the optimal coefficient vector \mathbf{w} can be directly obtained by numerically solving the system $\hat{A}\mathbf{w} = \hat{B}$. The matrix \hat{A} can have a very large condition number, so we have to carefully choose both the integration method and the basis functions, in order to obtain reliable results. To obtain the results presented in Sec. 4, we resorted to Gauss-Legendre integration formulas with $\lceil \frac{P}{m_1} \rceil \times \lceil \frac{P}{m_2} \rceil$ collocation points over each elementary rectangle (containing two simplices) of the domain S , and we used the ψ -basis introduced in [5], which is orthonormal with respect to the inner product directly related to the cost function F with $\lambda = 0$ (*i.e.*, to the cost function $F_1(\mathbf{w})$). The chosen value $P = 2000$ ensures reliable numerical results for many configuration of the vector \mathbf{m} .

3.2. Quality factor for the λ estimation

The fold bifurcation condition on the two branches T_1 and T_2 of the curve γ is

$$\frac{\partial h}{\partial x} = p_2 - 3x^2 = 0 \quad (8)$$

Then, in order to estimate the optimal value also for λ , we defined the following quality factor:

$$Q(\lambda) = \int_\gamma \left[\frac{\partial h}{\partial x} - \frac{\partial h_{PWL}}{\partial x} \right]^2 d\gamma = \int_\gamma \left[\frac{\partial h_{PWL}}{\partial x} \right]^2 d\gamma \quad (9)$$

Roughly speaking, such a function tends to its minimum when the solution – which has been determined to be close to h in the C^0 distance around the curve γ – tends to remain close to h in the C^1 distance as well. Owing to Eq. (8), the curve γ can be parameterized by x . Then, if we expand h_{PWL} in terms of the basis functions weighted by the optimal vector \mathbf{w} , the quality factor can be finally expressed as follows:

$$Q(\lambda) = \int_{x_-}^{x_+} \left[\sum_{k=1}^N w_k \frac{\partial \varphi_k(x, 3x^2)}{\partial x} \right]^2 \sqrt{1 + 36x^2} dx \quad (10)$$

where x_- and x_+ are proper integration boundaries.

The optimal λ corresponds to the absolute minimum of the quality factor.

3.3. Optimization algorithm

For fixed domain S and vector \mathbf{m} , to obtain the optimal λ , one could resort to *global* optimization procedures. As an alternative, one can explore a λ range $[\lambda_{min}, \lambda_{max}]$ for a fixed set of M values λ_i , by iterating the following three steps: (1) set $\lambda = \lambda_i$; (2) compute the matrix \hat{A} and the vector \hat{B} and find the optimal coefficient vector \mathbf{w} ; (3) compute the quality factor $Q(\lambda_i)$. Then, a more accurate estimate of the optimal value of λ can be estimated through *local* optimization procedures in a neighborhood of the λ_i corresponding to the minimum Q . Of course, the algorithm can also be iterated by varying the vector \mathbf{m} , to find the minimal simplicial partition that corresponds to a bifurcation diagram close to the original one. In the example proposed in the next section, we set $\lambda_{min} = 10^{-3}$ and $\lambda_{max} = 10^6$, and we iterated the three steps of the algorithm for 50 points per decade (in logarithmic scale).

4. Results and discussion

Once we have found the optimal values for both λ and the weights \mathbf{w} , we have to check the structural stability of the system (4) with h replaced by the obtained PWL approximation h_{PWL} . Actually, since we want to obtain a reliable bifurcation diagram by applying numerical continuation methods [7], that require smoothness of the vector field, h_{PWL} must be previously smoothed, as described in [1].

Figure 2(a) shows the equilibrium manifold for the approximated system obtained by setting $m_1 = 5$ and $m_2 = 24$ (*i.e.*, $N = 150$). The minimum of the quality factor Q ($Q_{min} \simeq 0.00963$) has been obtained for $\lambda = 3.39 \cdot 10^4$. For comparison, Fig. 2(b) shows the equilibrium manifold for the approximated system obtained for $\lambda = 0$, *i.e.*, by using the cost function $F_1(\mathbf{w})$.

Figures 3(a) and (b) show the bifurcation diagrams, that are the projections of the equilibrium manifolds of Figs. 2(a) and (b), respectively, on the parameter plane (p_1, p_2) . The comparison of the two approximations points out that in the figures (a) the equilibrium manifold is approximated reasonably well, and the bifurcation curves are very well approximated. In the figures (b), *vice versa*, the equilibrium manifold is approximated better, but the approximation of the bifurcation curves is less accurate.

5. Concluding remarks

The method presented in this paper has been applied to a very simple example of dynamical sys-

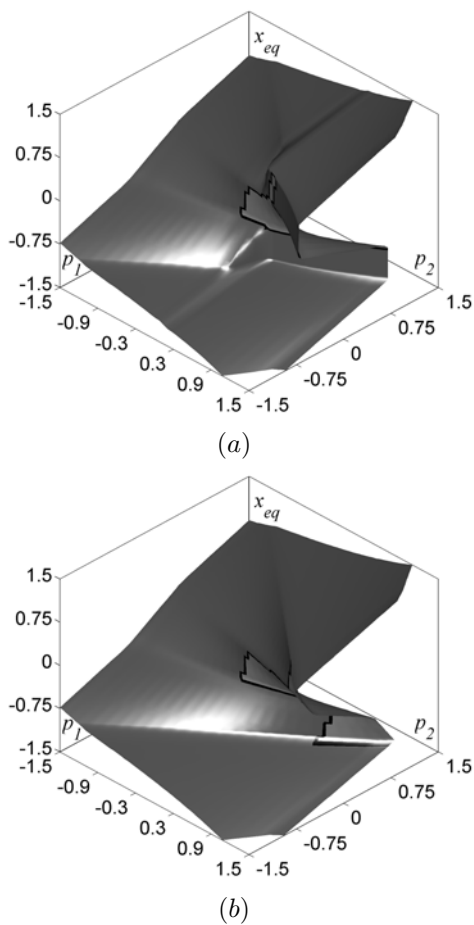


Figure 2: Approximations of the cusp normal form (4). Equilibrium manifolds $x_{eq}(p_1, p_2)$ obtained for $m_1 = 5$, $m_2 = 24$, and (a) $\lambda = 3.39 \cdot 10^4$, (b) $\lambda = 0$.

tem and has been partially fitted to the cusp normal form. Anyway, we point out that the method could be adapted, at least in principle, to other dynamical systems. The cost function terms weighted by the parameter λ can force the approximation to be particularly accurate in a neighborhood of either a bifurcation curve in the parameter space or an invariant set in the state space, for fixed parameters. For instance, the same line of reasoning could be applied to a dynamical system whose state portrait contains a limit cycle: it would be sufficient to add to the term $F_1(\mathbf{w})$ a line integral weighted by a parameter λ and evaluated over the limit cycle (which is an invariant curve), so as to force the PWL approximated dynamical system to have a limit cycle qualitatively and quantitatively similar to the original one. Of course, also a proper quality factor should be defined.

This very general method would allow one to obtain accurate PWL approximations of dynamical systems by focusing on the most important dynamical features of such systems. The proposed example is just a first

step towards this direction. The automatic selection of a partition ensuring a good compromise between the approximation accuracy and a reasonably low value of N is still an open problem.

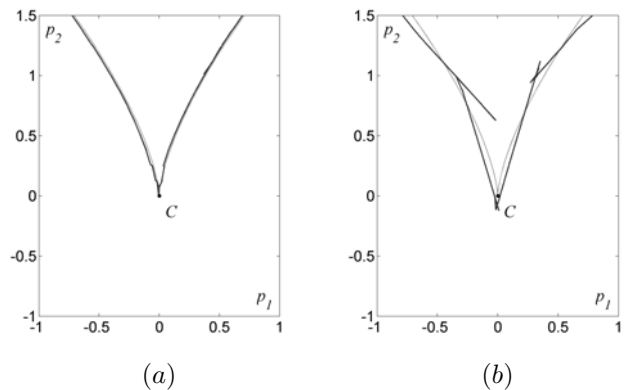


Figure 3: Approximations of the cusp normal form (4). Bifurcation curves obtained for $m_1 = 5$, $m_2 = 24$, and (a) $\lambda = 3.39 \cdot 10^4$, (b) $\lambda = 0$.

Acknowledgments

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