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Phase Transition in Adaptive Elementary Cellular Automata

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Abstract—We propose adaptive cellular automata that can independently change the rule set of each cell during evolution according to a metarule which refers the weighted mean of its own past states. Especially we take up the metarule that switches two elementary cellular automaton rule sets, rule 110 and rule 90, and investigate its behavior using entropy and power spectrum. The patterns change from rule 110-like to rule 90-like through periodic behavior as the threshold varies from zero to one. $1/f$ noise gradually emerges as the threshold is getting close to one and suddenly turns into white noise at the threshold equal to one.

1. Introduction

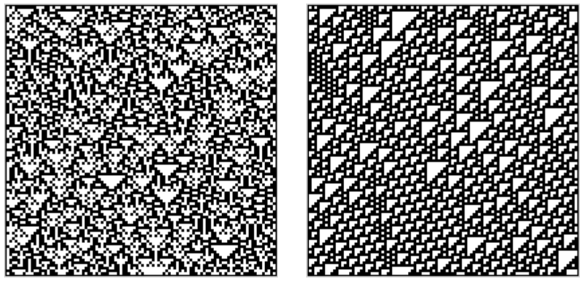
Cellular automaton (CA) is a n dimensional lattice in which finite automaton is attached in each lattice point called cell. The mapping from the set of lattice points to the finite set of cell states is called configuration. A bunch of cells referenced in state transition is called neighborhood and the transition takes place simultaneously in every cell according to a fixed transition function. For example in the case of the neighborhood consisting of the adjacent cells on both sides of one-dimensional array (n=1), transition function is given by

$$s_i^{(t+1)} = f(s_{i-1}^{(t)}, s_i^{(t)}, s_{i+1}^{(t)}), \quad (1)$$

where $s_i^{(t)}$ denotes the state of i th cell at time step t. By setting the set of states of cell as $\{0, 1\}$, we get elementary CA (ECA). ECA rule set is designated by the decimal number converted from 8 bit $f(1,1,1), f(1,1,0), \cdots, f(0,0,0)$.

It is known that CAs are classified into four classes by their behavior such as, null (class I), periodic (class II), chaotic (class III) and complex behavior (class IV) [1]. Especially class IV CAs are expected to be capable of supporting universal computation [2], [3]. Typical examples of space-time pattern of rule 90 (left) in class III and rule 110 (right) in class IV in ECA are shown Fig. 1.

While the rule set of conventional CAs is fixed during evolution, rule changing CAs has been proposed to solve some computational tasks such parity problem [4] or density classification problem [5]. These rule changing CAs can change rule set during evolution according to a schedule prepared in advance.



rule 90 rule 110

Figure 1: Space-time patterns of elementary cellular automaton rule 90 (left) and rule 110 (right).

However it seems reasonable to introduce another kind of rule changing CAs without any prior schedule as a model of primitive element that can not hold a complicated schedule in it due to its limited memory. In this article we propose rule changing CAs with metarule that can switch two ECA rule sets according to its history and investigate the change of the behavior as the parameters varies, especially focusing on phase transition because class IV CAs are expected to be at phase transition according to the hypothesis of "the edge of chaos" [2], [3].

In section 2 we propose adaptive CAs with binary metarule that switches two ECA rule sets according to cell's history. We focus on a binary metarule in section 3. Conclusions are given in section 4.

2. Binary Metarule

In this article we investigate CAs with metarule that can change rule sets. Generally speaking, metarules can be classified from several viewpoints mentioned below.

locality: If metarule refers solely to the sequence of the past states of its own cell, we call it *local*. If it refers to the sequences of the past states of other cells, it is called *non-local*.

planning: If metarule determines the rule set according to a plan made in advance, we call it *scheduled*. Otherwise we call it *dynamic*.

uniformity: If metarule applies a same rule set to all cells, we call it *homogeneous*. Otherwise we call it *heterogeneous*.

In this article we focus on local, dynamic, and heterogeneous metarules, especially changing rule sets according to its memory [6] and we call them 'adaptive' CAs.

The weighted mean $m_i^{(t)}$ of cell i at time step t is calculated by

$$m_i^{(t)} = \frac{\sum_{k=0}^{t} \alpha^{t-k} s_i^{(k)}}{\sum_{k=0}^{t} \alpha^{t-k}}, \quad (2)$$

where $\alpha \in (0, 1]$ characterizes memory term and is called memory factor. By defining $\omega_i(t)$ and $\Omega(t)$ as followings;

$$\omega_i(t) = \sum_{k=0}^{t} \alpha^{t-k} s_i^{(k)}, \ \Omega(t) = \sum_{k=0}^{t} \alpha^{t-k}, \quad (3)$$

we can express the weighted mean as $m_i^{(t)} = \omega_i(t)/\Omega(t)$ and $\omega_i(t)$, $\Omega(t)$ can be computed by the following recurrence relations;

$$\omega_i(t) = s_i^{(t)} + \alpha\omega_i(t-1), \quad \omega_i(0) = s_i^{(0)}, \quad (4)$$

$$\Omega(t) = \alpha^t + \Omega(t-1), \quad \Omega(0) = 1. \quad (5)$$

We introduce metarule M that determine the rule function $f_i^{(t)}$ of cell i at time step t according to the weighted mean $m_i^{(t)}$ such as;

$$f_i^{(t)} = M(m_i^{(t)}). \quad (6)$$

The state transition of cell i is given by

$$s_i^{(t+1)} = f_i^{(t)}(s_{i-1}^{(t)}, s_i^{(t)}, s_{i+1}^{(t)}). \quad (7)$$

In this paper we adopt metarule defined by

$$f_i^{(t)} = M(m_i^{(t)}) = \begin{cases} f_A & m_i^{(t)} > m_c \\ f_i^{(t-1)} & m_i^{(t)} = m_c \\ f_B & m_i^{(t)} < m_c, \end{cases} \quad (8)$$

where f_A and f_B represent single rule set and $m_c \in [0, 1]$ is the threshold of weighted mean to change the rule set. We call this kind of metarule 'binary metarule' and express it as f_A/f_B. Under the condition that $m_c = 0$ (1) and the initial rule f_A (f_B) is assigned to all cells, the metarule f_A/f_B can create the same evolution generated by the conventional CA with rule function f_A (f_B).

In this research f_A and f_B in Eq. (8) are chosen from ECA rule sets and we call that kind of metarule 'elementary binary metarule'.

3. Metarule 110/90

We study elementary binary metarule 110/90 that behaves like rule 110 at $m_c \approx 0$ and like rule 90 at $m_c \approx 1$. Throughout this article initial configuration and initial rule set are assigned randomly with equal probabilities and periodic boundary conditions are used.

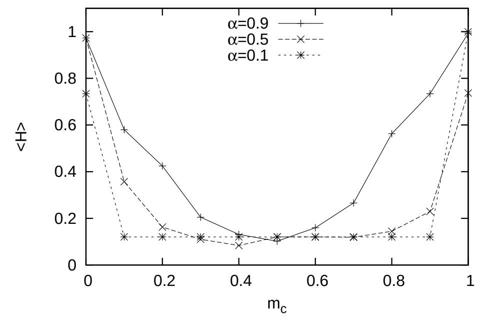


Figure 2: Average global entropy of metarule 110/90 with $\alpha = 0.9$, 0.5 and 0.1.

We compute entropy to investigate the behavior of metarule. The state $s_i^{(t)}$ is considered to be random variable and $P(X_i)$ can be estimated from the sequence $(s_i^{(0)}, s_i^{(1)}, \cdots, s_i^{(T-1)})$. The entropy $h(X_i) = -\sum_{i=0}^{1} P(X_i)\log P(X_i)$ of cell i is computed and we call the entropy averaged over all cells in the array 'global entropy' H. Figure 2 shows average global entropy $\langle H \rangle$ over the evolutions starting from five distinct initial configurations of 1,000 cells for 1,000 time steps for $\alpha = 0.9$, 0.5 and 0.1 with $\Delta m_c = 0.1$. The change in average global entropy $\langle H \rangle$ with varying m_c makes transitions conspicuous as α gets small.

Figure 3 shows the space-time patterns of metarule 110/90 with $\alpha = 0.1$ and various values of m_c. The patterns of rule set employed in the same evolutions are shown in Fig. 4 in which white square denotes the cell adopting rule 110 and black square the cell adopting rule 90. In the space-time pattern of $m_c = 0$ (top of Fig. 3) metarule 110/90 resembles rule 110 in behavior except that there are stable partition walls composed of rule 90 into which rule 110 can not penetrate (top of Fig. 4). At $m_c = 1$ (bottom of Fig. 3) metarule 110/90 closely resembles rule 90 in behavior except for the initial transient in which stable patterns are sparsely scattered (bottom of Fig. 4). The behavior exhibits like rule 110 and rule 90 as m_c is close to zero and one respectively and therefore $\langle H \rangle$ takes high value. In the range of $0.1 \leq m_c \leq 0.9$, the domains where rule 90 are dominant are compatible with the domains where rule 110 are dominant except for sparsely scattered oscillators as shown in the case of m_c=0.5 in the second from the top of Fig. 3 and Fig. 4. The periodic behavior brings about low value in $\langle H \rangle$ at intermediate value of m_c.

Next we perform spectral analysis of the evolutions of metarule 110/90 to confirm that phase transition occurs with varying m_c. The discrete Fourier transform of a time series $(s_i^{(t)})_{t=0}^{T-1}$ is given by

$$\hat{s}_i(f) = \frac{1}{T} \sum_{t=0}^{T-1} s_i^{(t)} \exp(-i\frac{2\pi t f}{T}). \quad (9)$$

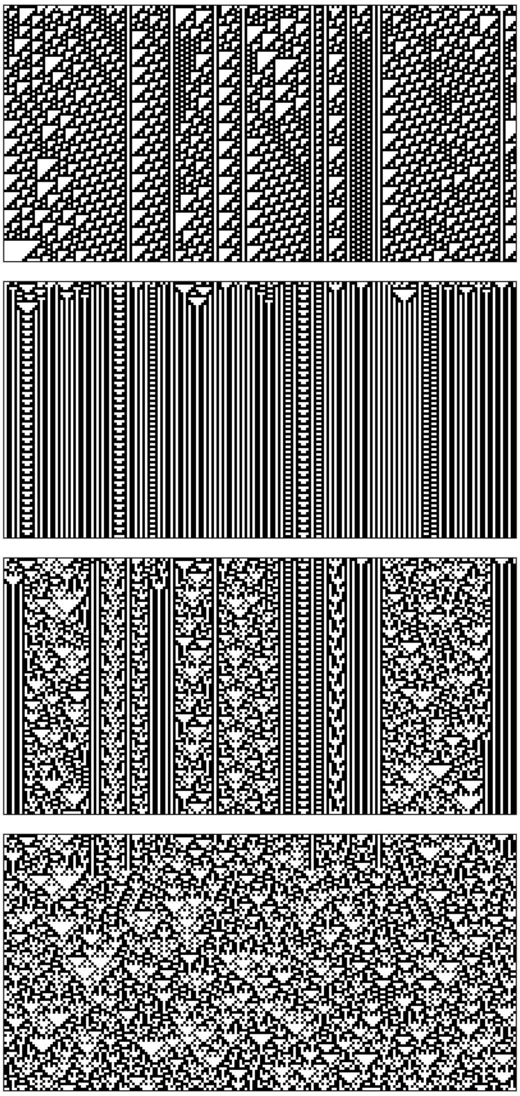


Figure 3: Space-time patterns of metarule 110/90 with α = 0.1 and m_c = 0, 0.5, 0.99999999 and 1 (from top to bottom).

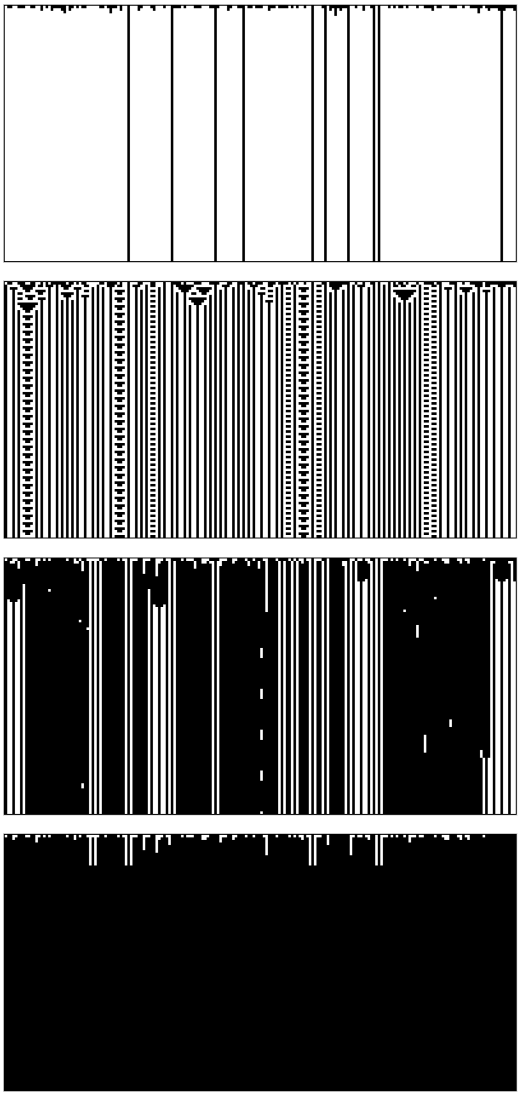


Figure 4: Rule patterns of the evolution shown on Fig. 3 with α = 0.1 and m_c = 0, 0.5, 0.99999999 and 1 (from top to bottom).

We define the power spectrum of CA as

$$S(f) = \frac{1}{N}\sum_i |\hat{s}_i(f)|^2, \tag{10}$$

where N denotes the total number of sites and the summation is taken in all sites. The least square fitting of power spectrum $S(f)$ by

$$\ln(S) = \alpha + \beta \ln(f), \tag{11}$$

from $f = 1$ to $f = f_b$ gives the exponent β.

Figure 5 shows the power spectra of metarule 110/90 with $\alpha = 0.1$ and various m_c. At $m_c = 0.0$ (top of Fig. 5), the power spectrum exhibits power law with exponent β = -1.23 in the range of frequencies $1 \leq f \leq 30$ since it is known that the evolution of rule 110 starting from random configuration exhibits power law [7]. In the range of $0.1 \leq m_c \leq 0.9$, the power spectra are characterized by sharp peaks caused by sparsely scattered oscillators as shown in the second from the top of Fig. 3. At $m_c = 1$, the behavior gets similar to rule 90 and the power spectrum exhibits white noise (bottom of Fig. 5).

However the transition from periodic phase at $m_c = 0.9$ to rule 90-like phase at $m_c = 1$ is not straightforward. As m_c is getting close to one, the power law arises as shown in the second from the bottom of Fig. 5. It has the exponent β = -1.36 in the range of $1 \leq f \leq 10$ at $m_c = 0.99999999$. The space-time pattern and the rule set pattern at this value of m_c are shown in the second from the bottom in Fig. 3 and Fig. 4 respectively. Rule 90 dominant areas are divided by partition walls composed of rule 110 cells. The wider the rule 90 dominant area is, the longer period it has. In other words, longer periodicity contributes more in power at low frequencies than shorter ones and that causes power law in power spectrum.

4. Conclusions

In this article we proposed CAs with binary metarule that can independently switch two ECA rule sets of each cell according to its weighted memory during evolution. We investigated the change of evolution in elementary binary metarule 110/90 as m_c varies by means of entropy and power spectra.

In metarule 110/90 the patterns change from rule 110-like to rule 90-like through periodic behavior varying m_c from zero to one. Spectral analysis has revealed that the characteristic of $1/f$ noise born in rule 110 is impaired as m_c increases from zero. And more strikingly $1/f$ noise gradually emerges once more as m_c is getting close to one and it suddenly turns into white noise at $m_c = 1$.

In the future plan, we are going to develop more complicated metarule that can switch ternary or more rule sets according to its memory to solve some tasks such as parity problem.

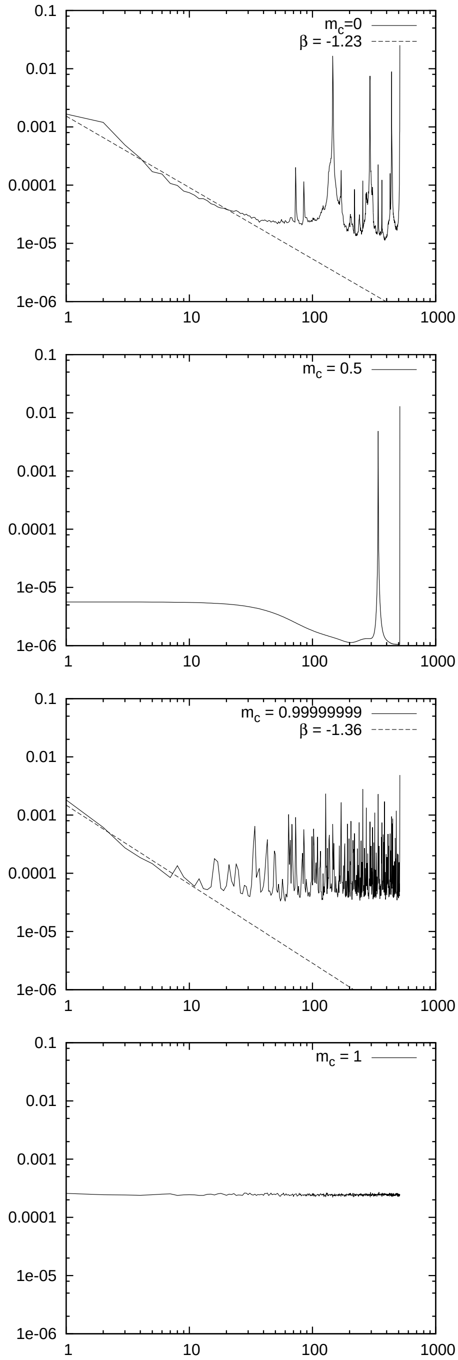


Figure 5: Power spectra of metarule 110/90 with $\alpha = 0.1$ and $m_c = 0$, 0.5, 0.99999999 and 1.0 (from top to bottom).

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