2017 International Symposium on Nonlinear Theory and Its Applications,
NOLTA2017, Cancun, Mexico, December 4-7, 2017

A Modified Algorithm and its Program for Obtaining Particular Solutions and Expansion Coefficients in P/T Petri Nets

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Abstract— P/T Petri nets are one kind of basic and useful model for discrete events and concurrent systems and firing count vectors for transitions are very important concepts when considering reachability problems which are the most important behavioral properties of Petri nets. To consider about the reachability problem, from M_0 to M_d (M_0 is an initial state called an initial marking, and also M_d is a destination state called a destination marking) are the fundamental problems of Petri nets. To solve such problems, we have some methods like using the cover ability(reachability) tree or using matrix equations. But the former method requires a huge amount of calculation in general. So, the latter method using matrix equations and reduction techniques has the advantage and is also better for using in computer calculations, because the method can utilize the algebraic equation properties of Petri nets. In this paper, we propose a modified algorithm of the Fourier-Motzkin method which is well known as a solution of the state equation for the reachability problem, and developing the program to use its algorithm. The solutions which could not be found by a conventional algorithm can be obtained by using the modified one. And also not only particular solutions and elementary T-invariants are obtained, but also the expansion coefficients of the nonnegative integer solution to represent state equations as $Ax = b$ can be obtained by the same program just by changing the input.

1. Introduction

A Petri net is a particular kind of directed graph, which has an initial state called the initial markings, M_0. A Petri net is a directed, weighted and bipartite graph where elements consist of two kinds of nodes, called places and transitions, and arcs which are either from a place to a transition or from a transition to a place. In graphical representation, places are drawn as circles, transitions as bars, and arks are labeled with their weights. The behavior of systems in Petri nets can be described by transitions firing. When transitions are fired, the marking which represents the number of tokens held by each place, is changed according to the transition firing rules. Such Petri nets are effectively used for modeling, analyzing, and verifying many discrete event systems[1].

In this paper, we show that both the particular solutions and the expansion coefficients can be obtained from the same program created based on the algorithm reported[2], by simply changing the input. We concern structural analysis based on linear algebra techniques and the state equation $Ax = b := M_d - M_0$, where M_0 and M_d are initial and destination marking vectors, respectively. All generators for T-invariants and all minimal inhomogeneous(i.e., particular) solutions are needed for discussing the feasibility of a group of firing count vectors, x, for the fixed $b := M_d - M_0$[3], where any firing count vector is expanded by means of T-invariant generators and particular solutions[4]. However, it is difficult, in general, to find the nonnegative rational/integer scalar expansion-coefficients. So, in this paper, We also consider how to find those coefficients, by using the same program created based on the algorithm of Fourier-Motzkin method. And furthermore, we would like to show a part of unobtainable particular solutions by using the conventional algorithm, could be found by using Modified Fourier-Motzkin method [5][6].

In section 2, preliminaries are given, and the method for finding expansion coefficients are described in section 3. In section 4, modified algorithm of Fourier-Motzkin method for finding the solutions which could not be obtained by conventional algorithm are described using an example. And section 5 is the conclusion of this paper.

2. Preliminaries

2.1. State Equation

If the destination marking M_d was assumed to be reachable from initial marking M_0 through the firing sequence as $\{t_1, t_2, \cdots, t_d\}$, the state equation can be expressed as

$$M_d = M_0 + A\sum_{k=1}^{d} t_k \tag{1}$$

and eq.(1) can be described like as eq.(2) when $A \in Z^{m\times n}$, $b = M_d - M_0 \in Z^{m\times 1}$, $x = \sum_{k=1}^{d} t_k \in Z_+^{n\times 1}$

$$Ax = b. \tag{2}$$

Then we can obtain the firing count vector x to solve the solutions of eq.(2), from initial marking $M_0 \in Z_+^{m\times 1}$ to destination marking $M_d \in Z_+^{m\times 1}$.

2.2. Fourier-Motzkin Method

The Fourier-Motzkin method is to obtain the set of all elementary vector solutions as the nonnegative integer solutions of $Ax = 0^{m\times 1}$. And the algorithm of the Fourier-Motzkin method is as follows [5][7].

<Algorithm of Fourier-Motzkin method>
Input: Incidence matrix $A \in Z^{m\times n}$, m, and n.
Output: The set of T-invariants including all minimal support T-invariants.
Initialization: The matrix B is constructed by adjoining the identity matrix $E^{n\times n}$ to the bottom of the incidence matrix $A \in Z^{m\times n}$, with $B = [A^T, E]^T \in Z^{(m+n)\times n}$.
Step0: $i = 1$.
Step1: Select the i-th row of B. If the i-th row has no nonzero element, then $i = i + 1$ and go to Step2. If the i-th row has at least one nonzero element, then go to Step3.
Step2: If $i \leqq m$ is satisfied, go to Step1, otherwise go to Step4.
Step3: Add to the matrix B all the columns which are linear combinations of pairs of columns of B and which annul the i-th row of B. And eliminate from B the columns in which the i-th element is nonzero. Now, let us call the new matrix as B again. Then set $i = i + 1$ and go to Step2.
Step4: Each column of the submatrix which is obtained by deleting the rows of the first to the m-th from B is a minimal nonnegative integer solution for $Ax = 0^{m\times 1}$.

But, this method can be applied to $Ax = 0$, and this means that obtained solutions are T-invariants. So, to obtain the particular solutions(firing count vectors), we need to make such changes to the eq.(2) considering the augmented incidence matrix as follows:

$$\widetilde{A} = [\ A \quad -b\ ] \in Z^{m\times(n+1)}. \tag{3}$$

then eq.(2) would be expressed by eq.(3) and augmented $\widetilde{x} \in Z^{n+1}$,

$$\widetilde{A}\widetilde{x} = 0. \tag{4}$$

Then, eq.(4) can be applied to the former algorithm.

3. Finding Expansion Coefficients for a Firing Count Vector by T-Invariants and Particular Solutions

§2.2 expressed how to obtain nonnegative solutions x of $Ax = b$ using the algorithm of the Fourier-Motzkin method. Finding expansion coefficients are useful for analyzing behavior verification of P/T Petri nets efficiently[8].

3.1. An Arbitrary Firing Count Vector by Means of T-Invariants and Particular Solutions

A firing count vector $x \in Z_+^{n\times 1}(x \in X)$ is expressed by using $u_i^{(4)} \in U_4$ = "the set of minimal support T-invariants" and $v_j^{(4)} \in V_4$ = "the set of fundamental particular solutions" as follows[5]:

$$x = \textstyle\sum_{i=1}^{l_4}\alpha_i^{(4)}u_i^{(4)} + \sum_{j=1}^{k_4}\beta_j^{(4)}v_j^{(4)}, \quad \sum_{j=1}^{k_4}\beta_j^{(4)} = 1, \tag{5}$$

where $l_4 = |U_4|$, $k_4 = |V_4|$, and $\alpha_i^{(4)}, \beta_j^{(4)} \in Q_+^{1\times 1}$.

We call eq.(5) as the level 4 expression in this paper. Moreover we have another expression for $x \in Z_+^{n\times 1}$ if we use $U_5 = \{U_4, U_5\backslash U_4\}$ = "the set of minimal T-invariants" and $V_5 = \{V_4, V_5\backslash V_4\}$ = "the set of minimal particular solutions" as follows[5]:

$$x = \textstyle\sum_{i=1}^{l_5}\alpha_i^{(5)}u_i^{(5)} + \sum_{j=1}^{k_5}\beta_j^{(5)}v_j^{(5)}, \tag{6}$$

where $\sum_{j=1}^{k_5}\beta_j^{(5)} = 1$, $l_5 = |U_5|$, $k_5 = |V_5|$, and $\alpha_i^{(5)}, \beta_j^{(5)} \in Z_+^{1\times 1}$. Then eq.(6) is rewritten as follows

$$x = \textstyle\sum_{i=1}^{l_5}\alpha_i^{(5)}u_i^{(5)} + v_j^{(5)}, \tag{7}$$

where $\beta_j^{(5)} = 1, v_j^{(5)} \in V_5$, and $\alpha_i^{(5)} \in Z_+^{1\times 1}$. We call eq.(6) or (7) as the level 5 expression in this paper. After that we discuss about the level 5 here , and also eq.(7) is rewritten as follows:

$$x = \sum_{i=1}^{l}\alpha_i u_i + v_j \tag{8}$$

where $\alpha_i \in Z_+^{1\times 1}$, $u_i \in U := \{\ u_i \in Z_+^{n\times 1}\ ;\ Ax = b$ T-invariants, and $i = 1, 2, \cdots, l\ \}$, $v_j \in V := \{\ v_j \in Z_+^{n\times 1}\ ;\ Ax = b$ particular solutions, $j = 1, 2, \cdots, k\ \}$, after here.

3.2. How to Find Expansion Coefficients

Eq.(8) means any nonnegative solutions (firing count vectors) of state equation eq.(2) can be obtained by the linear combinations of T-invariants and a particular solution.

When U, V, and $x \in Z_+^{n\times 1}$ are given, eq.(8) can be rewritten as follows:

$$\sum_{i=1}^{l}\alpha_i u_i = x - v_j \tag{9}$$

by transposition of v_j. And eq.(9) expresses

$$[\ u_1, u_2, \cdots, u_l\ ]\ \alpha = [\ x - v_j\ ]. \tag{10}$$

And on eq.(10),

$$[\ u_1, u_2, \cdots, u_l\ ] \to A', \ \ \alpha \to x', \ \ [\ x - v_j\ ] \to b'$$

are transposed, eq.(10) can be expressed as follows:

$$A'x' = b'. \tag{11}$$

This means that eq.(11) is the same type of equation as eq.(2). Then the same algorithm of the Fourier-Motzkin method expressed in §2.2 can be also applied to such problems as finding expansion coefficients for any reachable firing count vectors by T-invariants and a particular solution.

4. Finding Particular Solutions which could not be Obtained by using The Conventional Algorithm

Here, we would like to show an experimental example, that not all of particular solutions can be obtained by the algorithm described in §2.2 which is the conventional Fourier-Motzkin Method. And also the modified algorithm of Fourier-Motzkin Method can find some of particular solutions which are not found by conventional algorithm using the example.

4.1. Finding Particular Solutions by using The Conventional Algorithm

Using an example Fig.1, we would like to obtain some particular solutions by using the conventional algorithm of Fourier-Motzkin Method.

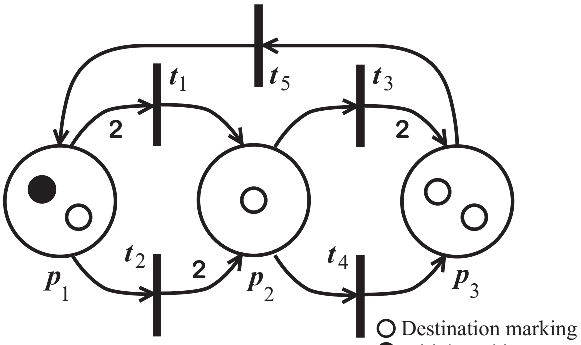

Fig.1 An example of a Petri net.

In this case, the incidence matrix of $A \in Z^{m\times n}$ is

$$A = \begin{bmatrix} -2 & -1 & 0 & 0 & 1 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 2 & 1 & -1 \end{bmatrix} \in Z^{3\times 5},$$

and the difference of marking $b \in Z^{m\times 1}$ from $M_0 \in Z_+^{m\times 1}$ to $M_d \in Z_+^{m\times 1}$ is

$$b = M_d - M_0 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \in Z^{3\times 1}.$$

Then the augmented matrix of A can be described as follows:

$$\widetilde{A} = \begin{bmatrix} -2 & -1 & 0 & 0 & 1 & 0 \\ 1 & 2 & -1 & -1 & 0 & -1 \\ 0 & 0 & 2 & 1 & -1 & -2 \end{bmatrix} \in Z^{3\times 6}$$

by eq.(3). And by the algorithm in §2.2, we can express the matrix B can be described as follows:

$$B = [\,\widetilde{A}\ \ E\,] \in Z^{(m+n+1)\times(n+1)}, \tag{12}$$

and from this eq.(12), T-invariants and particular solutions are obtained by using the algorithm of the conventional Fourier-Motzkin method as follows:

$$\begin{aligned} &u_1 = (\ 1\ \ 0\ \ 1\ \ 0\ \ 2\ )^T, \quad v_1 = (\ 0\ \ 3\ \ 0\ \ 5\ \ 3\ )^T, \\ &u_2 = (\ 1\ \ 1\ \ 0\ \ 3\ \ 3\ )^T, \end{aligned} \tag{13}$$

where $u_i \in U := \{u_i \in Z_+^{n\times 1}\}\ ;\ Ax = b$ T-invariants and $v_j \in V := \{v_j \in Z_+^{n\times 1}\}\ ;\ Ax = b$ particular solutions.

But here, when we think about a particular solution of

$$v_j = (\ 0\ \ 2\ \ 1\ \ 2\ \ 2\ )^T, \tag{14}$$

this solution is also the minimal vector like as eq.(13). Because not every element is smaller than the other solutions' elements. For example, the second element of eq.(14) **2** is smaller than v_1's second element of **3**, so eq.(14) can not be included by other obtained particular solution of v_1.

Hence, this means that we could not obtain all of the particular solutions by using the conventional algorithm of the Fourier-Motzkin Method.

4.2. Finding Particular Solutions by using a Modified Algorithm

All of the particular solutions could not be obtained by using conventional algorithm of the Fourier-Motzkin Method in §4.1. So, we guess the reason why, and this would be the algorithm of step 3 in §2.2 is not enough to find the all of combinations to make annul the i-th row of B.

In other words, annulment of the i-th row of B can be made by not only pairs of linear combinations, but more than 2 columns combinations. By using all of the combinations to make annul the i-th row of B, it would be able to find the new solutions which could not be found by using the conventional algorithm of Fourier-Motzkin Method.

Then the modified algorithm of the Fourier-Motzkin Method to improve this problem is as follows:

<Algorithm of Modified Fourier-Motzkin method>

Input: Incidence matrix $A \in Z^{m\times n}$, m, and n.

Output: All of minimal T-invariants.

Initialization: The matrix B is constructed by adjoining the identity matrix $E^{n\times n}$ to the bottom of the incidence matrix $A \in Z^{m\times n}$, where $B = [A^T, E]^T \in Z^{(m+n)\times n}$.

Step0: $i = 1$.

Step1: Select the i-th row of B. If the i-th row has no nonzero element, then $i = i + 1$ and go to Step2. If the i-th row has at least one nonzero element, then go to Step3.

Step2: If $i \leqq m$ is satisfied, go to Step1, otherwise go to Step8.

Step3: If the i-th row of B has at least one pair of positive and negative elements, go to Step4, otherwise go to Step7.

Step4: Aiming the i-th row of B(i.e., the old matrix), add directly the j-th column to the k-th column, where the (i, j) element is positive and the (i, k) element is negative. Apply the minimal vector criterion to the above new column vector and the column vectors each of which has the zero i-th element on the old matrix B. Adjoin all the remained columns after this criterion to the old matrix B. Then call this new matrix as B again. Then go to Step5.

Step5: If the i-th element of all the adjoined column vectors of the new matrix B is zero, go to Step7, otherwise go to

Step6.
Step6: Repeat Step4 to the matrix *B*. However, the (j, k) pair should be always new. Then, go to Step5.
Step7: Delete, from *B*, all the columns each of which has nonzero element on the *i*-th row of *B*. Now, let us call the new matrix as *B* again. Then set $i = i + 1$ and go to Step2.
Step8: Each column of the submatrix which is obtained by deleting the rows of the first to the *m*-th from *B* is a minimal nonnegative integer solution.

So, when we tried to obtain the solutions using this modified algorithm, finally, we could get the Matrix *B* of eq.(12), as shown in eq.(15).

Actually, the algorithm can not be done to the end. Eq.(15) is the calculation result to the middle, but on the

$$B = \left[\begin{array}{ccccccccccccc|cccccc}
 & & & & & & & & & \ast & & & & & & & & \ast\ast & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
3 & 2 & -1 & 1 & -2 & -5 & 2 & 1 & -2 & 0 & -3 & -1 & -4 & 1 & 0 & -1 & -2 & 0 & \ldots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & \ldots \\
2 & 1 & 1 & 0 & 0 & 0 & 3 & 2 & 2 & 1 & 1 & 0 & 0 & 4 & 3 & 2 & 1 & 0 & \ldots \\
0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 3 & 2 & 0 & 1 & 2 & 3 & 5 & \ldots \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & \ldots \\
0 & 0 & 1 & 0 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 1 & \ldots
\end{array}\right]. \tag{15}$$

18th column(marked ‘*’), this solution is already obtained by using the conventional algorithm in eq.(13). And paying attention to the 10th column(marked ‘**’), this is the solution just shown in eq.(14). So, the new solution which could not be found by using the conventional algorithm could be obtained by using the modified algorithm of the Fourier-Motzkin Method, even though the algorithm could not be done.

5. Conclusions

T-invariants and particular solutions for the state equation in P/T Petri nets can be obtained by using the Fourier-Motzkin Method. But some particular solutions could not be obtained by using the algorithm of the conventional Fourier-Motzkin Method. On the other hand, any firing count vector can be expressed by a combination of minimal support T-invariants and fundamental particular solutions. Also, to find such expansion coefficients representing the combination is useful for analyzing behavior verification of Petri nets efficiently.

In this paper, we showed that the expansion coefficients could be obtained by the same program using the algorithm of Fourier-Motzkin Method which can obtain minimal support T-invariants and fundamental particular solutions. And furthermore, we also showed that a part of unobtainable particular solutions using the conventional Fourier-Motzkin method, could be found by using modified algorithms Fourier-Motzkin method.

In future studies, we would like to improve the algorithm to let the calculation to the end, and also to obtain all of the unobtainable solutions.

Acknowledgments

This research was supported in part by a Grant-in Aid for Scientific Researches(C) Grant Numbers 16K01138, 24501221 Japan Society for the Promotion of Science(JSPS).

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