

Propagation sustained by perturbation in a nonlinear diffusive medium

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Abstract—We show numerically and theoretically that a Nagumo lattice initially in propagation failure conditions can sustain information propagation thanks to an additive perturbation.

1. Introduction

In the last few years, a growing interest has been devoted to nonlinear dissipative or reaction diffusion systems in many fields, like physics, chemistry, or biology [1]. Among these systems is the Nagumo model expressed here in the discrete case [2]

$$\frac{du_n}{dt} = D(u_{n+1} + u_{n-1} - 2u_n) + f(u_n), \quad (1)$$

where D represents the diffusion coefficient and $f(u) = -u(u-a)(u-1)$ the cubic nonlinearity.

For weak couplings D below a critical value D^* propagation fails, which may induce fatal consequences in the field of neuro- and cardiophysiology. Until now, most of the studies have been devoted to the determination of D^* in undisturbed media [3, 4, 5, 6]. However, myelinated nerve fibers as well as cardiac tissues modeled by these reaction diffusion equations are rather inhomogeneous than homogeneous and often submitted to perturbations whether random or not. Therefore, taking into account inhomogeneities [7, 8, 9, 10, 11, 12] as well as constant or random perturbations can provides new conditions for propagation failure.

First, we present the mechanical analogy of the Nagumo equation (1). Especially, the origin of propagation failure is briefly discussed. Then, in the third section, we investigate the effect of an additive perturbation in a homogeneous Nagumo lattice initially in propagation failure condition. Using the previous mechanical analogy, we predict the behavior of the system, and show the possibility for the medium to sustain information propagation in a given range of perturbation.

2. The mechanical analogy

From a mechanical point of view, eq. (1) models an overdamped chain of harmonically coupled particles of mass M

submitted to a cubic force f and whose inertia term is neglected (fig 1.(a)).

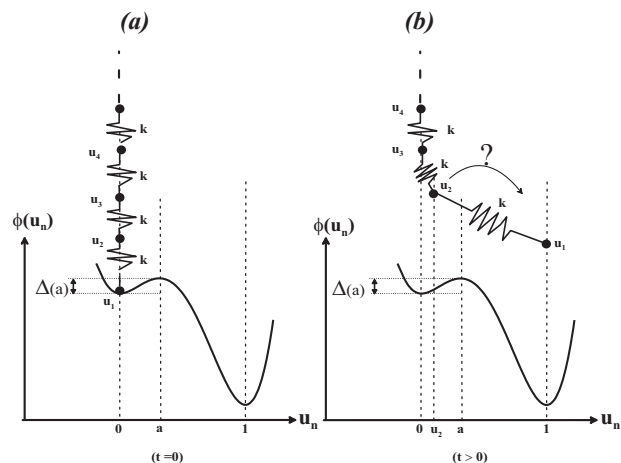


Figure 1: The propagation failure mechanism. (a): at $t=0$, overdamped chain of harmonically coupled particles of masse M experiencing the double well potential ϕ defined by $f(u) = -\frac{d\phi}{du}$. (b): chain at $t > 0$, the first particle has crossed the potential barrier $\Delta(a)$ thanks to an external forcing and attempt to pull the second particle down in its fall by mean of its coupling spring.

The diffusion coefficient D is related to the strength k of the springs and the friction term λ by $D = k/\lambda$, whereas the force $f(u) = -\frac{d\phi}{du}$ derives from the double well on site potential ϕ . Moreover, the position of the potential extrema, that is the roots $0, a, \text{ and } 1$ of the cubic force f , correspond to the steady states of the system.

To understand the propagation failure mechanism, let us consider the case of weak couplings and of a nonlinearity threshold $a < 0.5$.

Initially all particles of the chain are located at the position 0 (fig. 1.a). To initiate a kink, an external forcing allows the first particle to cross the potential barrier in $u = a$ and to fall in the right well at the position $u = 1$ (figure 1.(b)). Thanks to the spring coupling the first particle to the second one, but despite the second spring, the second particle attempts to cross the potential barrier with height $\Delta(a) = -\frac{a^4}{12} + \frac{a^3}{6}$

(figure 1.(b)).

According to the value of the resulting force applied to the second particle by the two springs compared to the nonlinear force f between $[0, a]$, two behaviors may occur:

- (i) If the resulting force is sufficiently important to allow the second particle to cross the potential barrier $\Delta(a)$, then this particle falls in the right well and pulls the next particle down in its fall. It gives then rise to a propagative kink, whose velocity increases versus the coupling and as the barrier decreases (namely, as a decreases).
- (ii) Else, if the resulting force does not exceed a critical value, (i.e. if $D < D^*(a)$), the second particle cannot cross the potential barrier and thus stays pinned at a position u in $[0, a]$: it is the well known propagation failure effect.

In summary, the mechanical model associated to eq. (1) reveals that the characteristics of the medium are given by the coupling D and the potential barrier $\Delta(a)$ (i.e. a), which define whether the medium is in propagation conditions or not.

3. Theoretical study

In this section, we investigate how the previous propagation conditions are modified by an additive perturbation. We thus consider the equation of a Nagumo chain initially in propagation failure conditions and submitted to a constant perturbation ϵ

$$\frac{du_n}{dt} = D(u_{n+1} + u_{n-1} - 2u_n) + f(u_n) + \epsilon. \quad (2)$$

Here, the coupling D is chosen below the critical value D^* such that, without perturbation ($\epsilon = 0$), the medium is unable to sustain kink propagation. We investigate then the effect of a constant perturbation ϵ on the pinning or depinning conditions.

In fact, one can include the perturbation in the nonlinear force f to define the following effective force:

$$F(u, \epsilon) = f(u) + \epsilon = -(u - \theta)(u - \alpha)(u - \beta), \quad (3)$$

whose new roots $\theta(\epsilon) < \alpha(\epsilon) < \beta(\epsilon)$ correspond to the extrema of the effective on-site potential

$$\begin{aligned} \Phi(u, \epsilon) &= \phi(u) - \epsilon u \\ &= \frac{u^4}{4} - \frac{a+1}{3}u^3 + \frac{a}{2}u^2 - \epsilon u. \end{aligned} \quad (4)$$

Note that, the potential barrier height $\Delta(\epsilon, a) = \Phi(\alpha) - \Phi(\theta)$, as well as the roots θ, α, β depend on the additive perturbation ϵ . As represented in figure 2.(a), the effective force F appears as the force f after a vertical translation of a quantity ϵ , and the effective potential barrier height decreases for $\epsilon > 0$ or increases for $\epsilon < 0$ (Fig. 2.(b)).

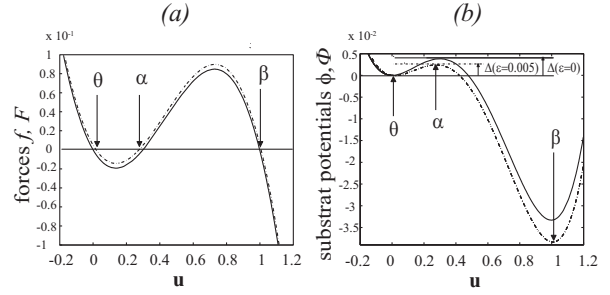


Figure 2: Effect of the perturbation on the force and the potential. Parameters: $\epsilon = 5 \times 10^{-3}$, $a = 0.3$. (a) : The effective force F (dotted line) and the initial force f (solid line). (b) : on-site potential ϕ without perturbation ($\epsilon = 0$, solid line), and effective potential Φ (dotted line).

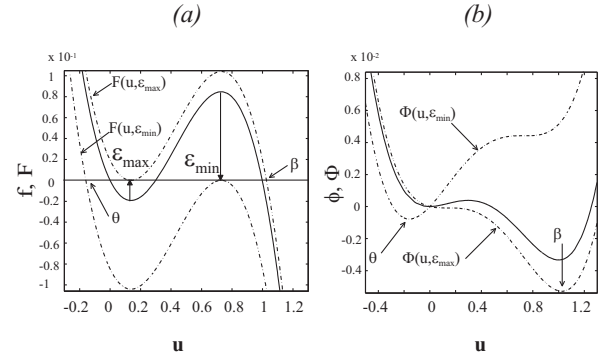


Figure 3: Existence of the Bistability. (a) force without perturbation (solid line) and effective forces (dotted line) $F(u, \epsilon_{max})$, $F(u, \epsilon_{min})$ obtained for the additive perturbations $\epsilon_{max} = 0.0192$ and $\epsilon_{min} = -0.0847$. (b) : corresponding on-site potentials. Nonlinearity threshold: $a = 0.3$.

Moreover, if the perturbation is too important, as exhibited in figures 3.(a) and (b), the effective potential is no more a double well potential and the system loses its bistability. Consequently, propagative or stationary kink solutions cease to exist if the perturbation is outside the range $]\epsilon_{min}, \epsilon_{max}[$.

ϵ_{min} corresponds to a negative perturbation for which the only stable steady states is θ , while ϵ_{max} is a positive perturbation inducing the monostability of the system with single stable steady state β .

3.1. Effects of the perturbation on propagation failure

According to the sign of ϵ , the barrier is reduced or increased changing the propagation failure conditions, so we will consider separately the case of a positive perturbation and the case of a negative one.

- (i) $\epsilon > 0$:

As the effective potential barrier height $\Delta(\epsilon) = \Phi(\theta) - \Phi(\alpha)$ decreases versus ϵ , for a positive perturbation ϵ

exceeding a critical value ϵ_{inf}^* , the barrier height $\Delta(\epsilon)$ can be so reduced that, for the considering coupling value D , the system is no more in propagation failure conditions and allows the propagation of a kink.

(ii) $\epsilon < 0$:

By contrast, if the negative perturbation decreases, since the effective potential barrier height increases, the medium first remains in propagation failure conditions. However, if ϵ is sufficiently small, the symmetry of the potential is reversed and the potential barrier height becomes $\Delta(\epsilon) = \Phi(\beta) - \Phi(\alpha)$ (see fig. 3.(b) in the limit case $\epsilon = \epsilon_{min}$).

Therefore, for a negative perturbations ϵ below a critical value ϵ_{sup}^* , the effective barrier reduction allow once again a travelling kink with a negative velocity since the potential symmetry is reversed.

These four predicted behaviors have been numerically confirmed and reported in figure 4:

- (i) For $\epsilon \in]\epsilon_{min}; \epsilon_{sup}^*[$, the propagation of a kink with a negative velocity is possible (region I).
- (ii) For $\epsilon \in]\epsilon_{sup}^*; \epsilon_{inf}^*[$ the system remains in propagation failure conditions (region II).
- (iii) For $\epsilon \in]\epsilon_{inf}^*; \epsilon_{max}[$ a kink spreads with a growing positive velocity versus epsilon (region III).
- (iiii) For ϵ outside $]\epsilon_{min}; \epsilon_{max}[$, kink ceases to exist (forbidden range in hatched lines since the system loses its bistability).

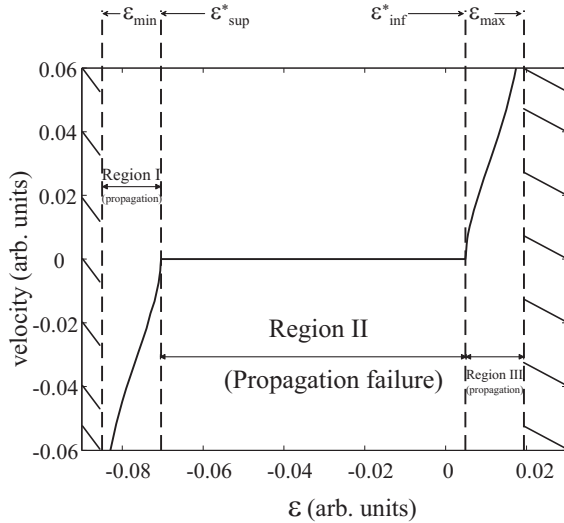


Figure 4: Behavior of the system versus the nonlinearity threshold a and the perturbation ϵ .

3.2. Minimum perturbation allowing propagation

Since the medium is initially in propagation failure conditions, the coupling is sufficiently weak to assume, as in [5, 6], that only site 2 experiences the effective nonlinearity while sites 1 and 3 are close enough to the effective potential minima $u_1 \simeq \beta$ and $u_3 \simeq \theta$. To find the largest perturbation ϵ_{inf}^* for which the system remains in propagation failure, we are lead to determine the position $u_2 = u \in]\theta; \alpha[$ of the pinned particle obeying to

$$D(\theta + \beta - 2u) + f(u) + \epsilon = 0. \quad (5)$$

Using the procedure exposed in [6], we replace the cubic nonlinearity in $]\theta; \alpha[$ by its parabolic approximation $f(u) \simeq g(u) = (1 - a/2)u(u - a)$. Thus, in the range $]\theta; \alpha[$, the effective force can be estimated by

$$F(u, \epsilon) \simeq (1 - a/2)(u - a)u + \epsilon. \quad (6)$$

For weak perturbations, $\theta(\epsilon)$ and $\alpha(\epsilon)$ can be approximated by the roots of (6)

$$\begin{aligned} \theta(\epsilon) &\simeq \frac{a}{2} - \frac{\sqrt{4a^2 - 4a^3 + a^4 + 8a\epsilon - 16\epsilon}}{2 - a} \\ \alpha(\epsilon) &\simeq \frac{a}{2} + \frac{\sqrt{4a^2 - 4a^3 + a^4 + 8a\epsilon - 16\epsilon}}{2 - a}, \end{aligned} \quad (7)$$

while at first order $\beta(\epsilon)$ writes

$$\beta(\epsilon) \simeq 1 + \epsilon. \quad (8)$$

Substituting the approximations of β , F , and θ in eq. (5) provides a second order equation in u whose discriminant $\delta(D, \epsilon)$ must be positive to ensure the existence of a unique solution for the position $u \in]\theta; \alpha[$ of the pinned particle (see figure 1.b). The critical perturbation ϵ_{inf}^* is then given by

$$\begin{aligned} \epsilon_{inf}^*(D, a) &= \frac{1}{32(2D + 1 + D^2)(a - 2)} \\ &\times \left(-64D^2a + 16a^3 - 16a^2 - 4a^4 \right. \\ &+ 16D^2a^2 + 64D + 16Da^3 + 32D^2 \\ &\left. - 64Da - 64D^3 - 4a^4D - 16\sqrt{q} \right), \\ \text{with } q &= -16D^5 + 8D^4a^2 \\ &+ D^3a^4 - 4D^4a^3 - 16D^4a + D^4a^4 \\ &- 4D^3a^3 + 4D^4 + 16D^3 - 16D^3a \\ &+ 8D^3a^2. \end{aligned} \quad (9)$$

3.3. Numerical results:

To valid our theoretical expression of ϵ_{inf}^* , we have performed a numerical simulation of system (2) using a fourth order Runge-Kutta algorithm with an integrating time step

$dt = 10^{-2}$. First, for $t < t_0$, the system is simulated for $\epsilon = 0$ and a given coupling D below the critical value $D^*(a = 0.3) = 0.0286$, so that a kink initiated from a Heaviside-type distribution in a lattice of 50 cells is pinned. Once the pinning is numerically realized, for $t > t_0$ a constant perturbation ϵ is added, and we analyze versus ϵ whether the kink spreads or not in the whole lattice.

Performing with the same procedure several simulations with different values of ϵ , by dichotomy, we numerically estimate the critical perturbation ϵ_{inf}^* beyond which kink propagation takes place.

The numerical results obtained for different worths of the coupling $D < D^*(a = 0.3) = 0.0286$ are plotted in figure 5 ((o) signs) and match with a perfect agreement the theoretical expression (9) (solid line (a)). The curve (a) defines two regions in the parameters plane: at the right where propagation is possible and at the left where the system remains in propagation failure.

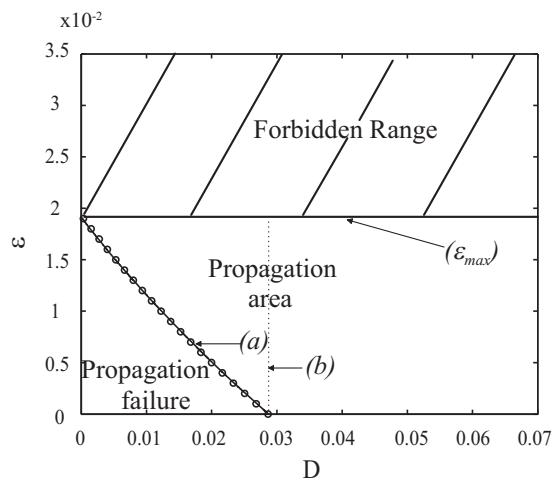


Figure 5: Behavior of the systems versus the perturbation and the coupling. Threshold nonlinearity $a = 0.3$. (a): Critical perturbation ϵ_{inf}^* allowing kink propagation in a medium initially in propagation failure versus the coupling D ; the solid line is obtained with the theoretical expression (9), while the (o) signs corresponds to numerical simulations.

To compare with the unperturbed case, we have drawn in dotted line (curve (b)) the critical coupling $D^*(a = 0.3) = 0.0286$ inducing propagation failure in a non-perturbed medium. It reveals then clearly that applying an appropriate perturbation significantly reduces the propagation failure effects.

4. Conclusion

We have numerically and theoretically shown that except if the coupling is null, there exists a range of perturbations allowing kink propagation in a medium initially in

propagation failure regime. As reported in [13], this effect can be extended to the case of random perturbation where noise can sustain kink propagation in a medium initially in propagation failure conditions. Moreover, our study could be useful in better understanding the behavior of inhomogeneous nerve or cardiac tissues where structural inhomogeneities hinder information transmission.

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