PROCEEDINGS OF ISAP '92, SAPPORO, JAPAN

HYBRID PARABOLIC EQUATION FOR NONUNIFORM PLASMA WAVEGUIDE

I.O.Novikova, A.V.Popov

Institute of terrestrial magnetism, ionosphere and radio wave propagation, Academy of Science IZMIRAN, 142092 Troitsk, Moscow region, Russia

Most of the features of VLF propagation are well known. On the other hand, calculation of the mode conversion over nonuniform parts of the earth-ionosphere waveguide is a difficult analytic problem. We propose a new approximation for the low-order modes based on a modification of the Leontovich-Fock parabolic equation [1]. It takes into account refraction and absorption in the vicinity of the effective height of reflection and allows to avoid uncertainty in determining the impedance of the curvilinear upper boundary. The approximate solution is constructed as asymptotics in small parameters inherent in the problem: short-wave parameter λ/H - ratio of the wavelength to the characteristic height H of the waveguide, smoothness parameter H/L where H is characteristic scale of longitudinal nonuniformity and angular distance L/a where a is the earth radius. More accurate choice of the main oscillating factor compared with [1] leads to better description of diffraction effects combined with geometric optics of reflection in the upper part of the waveguide.

We consider the scalar wave equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{\varepsilon}\frac{\partial U}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\varphi}\left(\frac{1}{\varepsilon}\frac{\partial U}{\partial\varphi}\right)+k^2U=0, \qquad (1)$$

resulting from Maxwell equations for TM waves in a dielectric medium with nonuniform complex permittivity ε depending on the radial coordinate r and the azimuthal angle φ . The electric field of a vertical dipole can be expressed approximately as

$$E_z = -\,300\sqrt{\frac{ia\lambda N}{r\sin\varphi}}\;U \quad \text{mv/m} \qquad (2)$$

where N kw is transmitted power and $U(r,\varphi)$ - the solution of Eq.(1) having proper singularity at the initial point.

In order to construct an asymptotic solution of Eq.(1) we put $\varepsilon = \varepsilon(x,z)$ where $x = a\varphi/L$ and $z = (r-a)/H$ are normalized dimensionless variables. It is supposed that $\nu=H/L$ and $\mu=L/a$ are small parameters and that the Fresnel parameter $\sigma = kH^2/L$ is of the order of unity (it is true for VLF propagation with $k = 2\pi/\lambda \sim 0.3\ \text{km}^{-1}$, $H \sim 60\text{-}90$ km, $L \sim 500\text{-}1000$ km). We choose the following Ansatz

$$U(z,\varphi) = w(x,z)\sqrt{\frac{\varepsilon}{r}}\;e^{i\sigma\Phi} \qquad (3)$$

where $\Phi(x,z) = x + \nu\,\psi(x,z)$ is an average eikonal of the rays starting from the earth surface with small elevation angles. After substituting the Ansatz into Eq.(1) and neglecting the terms of order $O(\nu)$ and $O(\mu)$ we obtain the following equation for the smoothly varying amplitude function $w(x,z)$:

$$i\frac{\sigma}{\nu}\left[2\psi_z\frac{\partial w}{\partial z}+\left(\psi_{zz}+2i\sigma\psi_x\right)w\right]+2i\sigma\frac{\partial w}{\partial x}+\frac{\partial^2 w}{\partial z^2}+$$
$$+\left[\frac{1}{2}S+\sigma^2\left(2\frac{\mu}{\nu}z-\psi_x^2\right)\right]w=0 \qquad (4)$$

Here the complex function ψ is determined as $\psi(x,z)=\int\sqrt{\varepsilon-1}\,dz$; φ_x, ψ_z, ψ_{zz} are its partial derivatives and

$$S(x,z)=\frac{\varepsilon_{zz}}{\varepsilon}-\frac{3}{2}\frac{\varepsilon_z^2}{\varepsilon^2} \qquad (5)$$

is Schwartzian of the function $\int\varepsilon\,dz$.
Let us discuss the meaning of separate terms in Eq.(4). The first-order differential operator in square brackets corresponds to the WKB approximation. It is very small in the neutral atmosphere because the complex dielectric permittivity has the form

$$\varepsilon\approx 1+i\alpha(x,z) \qquad (6)$$

with conductivity $\alpha(x,z)$ negligible in the absence of ionization. For $\alpha(x,z)$ is growing with height, the WKB term becomes dominating in the upper part of the earth-ionosphere waveguide. The "parabolic" operator $2i\sigma\,\partial/\partial x+\partial^2/\partial z^2$ describes the transversal diffusion of the wave amplitude. Coefficient $S(x,z)/2$ takes into account refraction due to the nonuniform content of the waveguide and the term $2\sigma^2\mu z/\nu$ reflects the influence of the earth sphericity.

When the approximate differential equation is found one can return to the physical variables (distance along the earths surface $x=a\varphi$ and height $z=r-a$) just by putting formally $H=L=1$. Then Eq.(4) takes the form

$$2ik\left(\frac{\partial w}{\partial x}+\psi_z\frac{\partial w}{\partial z}+\frac{1}{2}\psi_{zz}w\right)+\frac{\partial^2 w}{\partial z^2}+$$
$$+\left[\frac{1}{2}S+k^2\left(2\frac{z}{a}-\psi_x^2-2\psi_x\right)\right]w=0. \qquad (7)$$

equivalent to the Leontovich - Fock parabolic equation [1]

$$2ik\frac{\partial w}{\partial x}+\frac{\partial^2 w}{\partial z^2}+2k^2\frac{z}{a}w=0 \qquad (8)$$

in the lower part of the waveguide where $\varepsilon(x,z)\equiv 1$, $\psi(x,z)\equiv 0$, $S(x,z)\equiv 0$.

The discrepancy between Eqs.(7) and (8) is growing with height, and, above the region of essential refraction, the former degenerates into an approximate transport equation of geometric optics

$$\frac{\partial w}{\partial z}+\left[\frac{\psi_{zz}}{2\psi_z}-\frac{ik}{\psi_z}\left(\frac{z}{a}-\psi_x-\frac{1}{2}\psi_x^2+\frac{1}{4k^2}S\right)\right]w=0. \qquad (9)$$

It can be seen easily that, even for rapidly (e.g. exponentially) growing conductivity $\alpha(x,z)$, the coefficient in the square brackets is small and the wave amplitude $w(x,z)$ remains slowly varying. This chooses the solutions of the Eq.(1) vanishing inside the ionospheric layer in accordance with Eq.(2). Therefore, Eq.(9) serves as an inherent boundary condition for our parabolic equation (7). Since the first-order equation (9) holds in some boundary layer the solution is practically independent on arbitrary choice of the upper boundary of the waveguide. It is an obvious advantage compared with the usual impedance approach.

After supplementing Eq.(9) by the Leontovich condition

$$\frac{\partial w}{\partial z} + ikg_0 w = 0 \tag{10}$$

on the earth surface and the initial condition

$$w(+0,z) = 2\,\delta(z). \tag{11}$$

corresponding to the field of the vertical dipole Eq.(2), we obtain a complete boundary problem for the hybrid parabolic equation (7). It is very convenient for numerical solution because the wave amplitude $w(x,z)$ varies much slower than the primary function $U(r,\varphi)$ to be found.

Numerical implementation of the finite difference method makes no difficulties. The only problem is the singularity of the solution at the initial point. In order to overcome this complication we transfer the initial condition into an intermediate cross-section $x = x_0$ of the waveguide using an analog of the reciprocity theorem. As follows from Eq.(7), the integral relation

$$\int_0^\infty w(x,z)\, v_m(x,z)\, e^{2ik\psi(x,z)}\, dz = 2\, v_m(0,0) \tag{12}$$

exists between the function $w(x,z)$ and an arbitrary solution of the conjugated problem $v_m(x,z)$. We represent the solution in the cross-section $x=x_0$ as a series in some complete set of functions $f_m(z)$:

$$w(x_0,z) = \sum_m a_m\, f_m(z) \tag{13}$$

After solving the conjugated problem with initial conditions $v_m(x_0,z)= f_m(z)$ and determining the coefficients a_m from Eq.(12), we obtain a regular initial condition for further numerical integration of the parabolic equation (7).

Here, we demonstrate an example of field calculation for a model distribution of conductivity $\alpha(x,z)$ simulating the night-to-day transition. The lines of equal conductivity are depicted at Fig.1. The height drop of 6 km has been chosen with the longitudinal scale about 1000 km. The initial condition (13) has been posed in the cross-section x_0= 500 km, with four solutions of the conjugated problem involved. The calculated amplitude distribution along the earth surface is represented at Fig.2. (solid line). For comparison, the field amplitude for the uniform waveguide with the same initial conditions is plotted with the dashed line. The difference allows to distinguish the sunrise effect on the background of the usual interference fading.

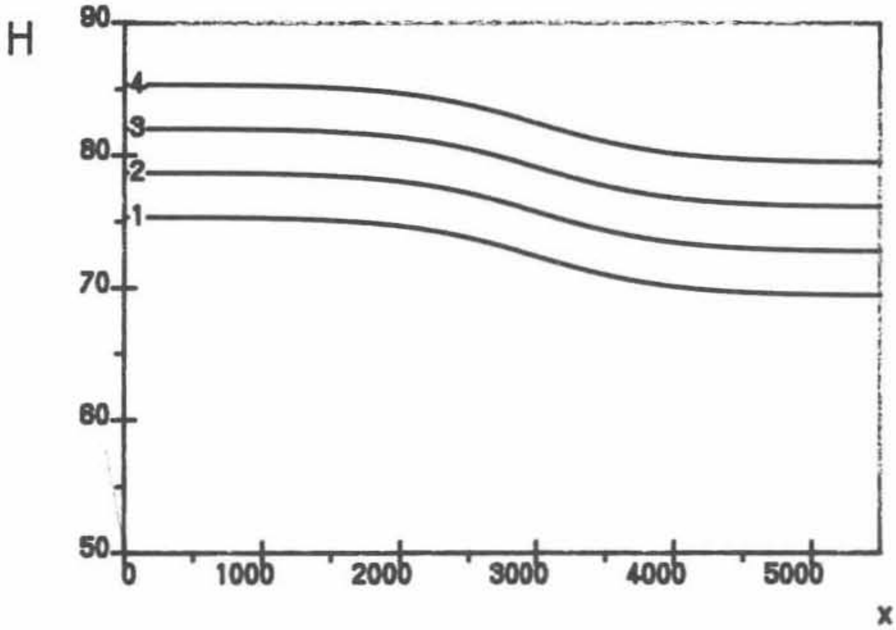

Fig.1. Model conductivity distribution (lines of equal values $\alpha(x,z) = 10^n$, $n = 1, 2, 3, 4,$)

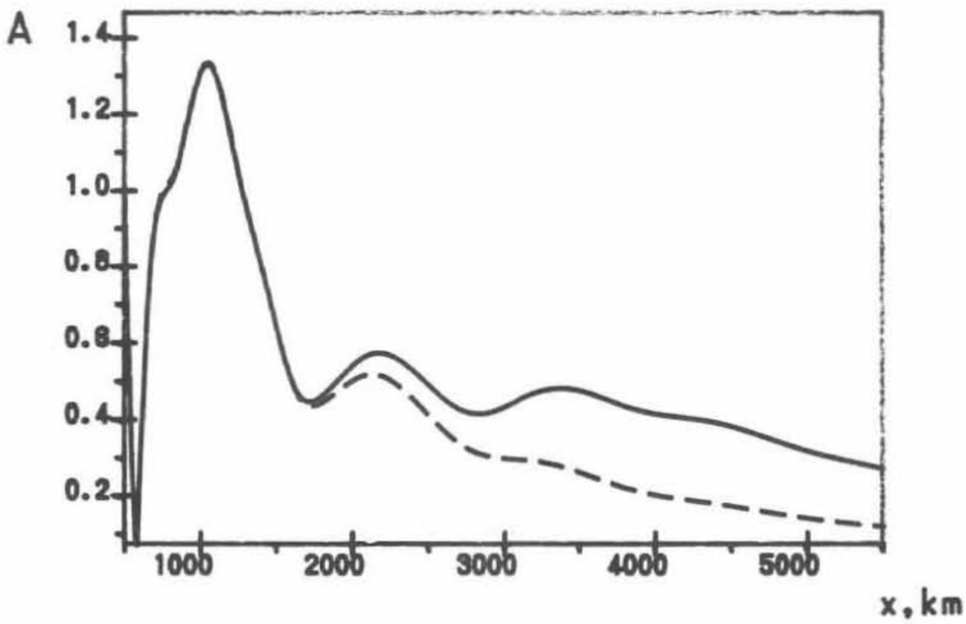

Fig.2 Amplitude distribution along the earth surface (*night-to-day transition* - solid line, *uniform waveguide* - dashed line)

References

[1] V.A. Fock, *Electromagnetic Diffraction and Propagation Problems*, Pergamon, London (1965)