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SIMULTANEOUS RECONSTRUCTION OF PHASE VELOCITY AND DISSIPATION COEFFICIENT IN A STRATIFIED HALF-SPACE USING 3-D REFLECTIVITY

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Consider the following hyperbolic equation

$$\frac{1}{c(z)^2}\frac{\partial^2}{\partial t^2}u - (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})u + b(z)\frac{\partial}{\partial t}u = 0, \quad (x, y, z) \in \mathcal{R}^3, \tag{1}$$

in a stratified half-space $z > 0$ where the parameters $c(z)$ (phase velocity) and $b(z)$ (dissipation coefficient) vary with the depth z. The sources, with compact spatial support, are located in the upper half-space $z < 0$ which is homogeneous and non-dissipative. We assume that the incident fields will not reach the surface $z = 0$ until the time $t = 0$, i.e., we have the initial conditions

$$u(x, y, z, 0) = 0, \qquad u_t(x, y, z, 0) = 0, \quad \text{for } z > 0. \tag{2}$$

In the inverse problem, we assume that only reflection data in the upper homogeneous half-space are measurable and we wish to achieve a simultaneous reconstruction of the phase velocity $c(z)$ and the dissipation coefficient $b(z)$.

The present inverse approach is based upon the concept of wave splitting, which is associated with the factorization of wave equation [1]−[5]. Wave splitting refers to the decomposition of the total wave into up- and down-going waves with respect to parallel planes in the inhomogeneous medium. Thus, we introduce the following wave splitting

$$u^{\pm} = (1/2)[u \mp \mathcal{K}u_z], \tag{3}$$

where $\mathcal{K}$ and its inverse $\mathcal{K}^{-1}$ satisfy

$$\mathcal{K}^{-1} = (\frac{1}{c(z)^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2})\mathcal{K}. \tag{4}$$

The explicit form of the splitting operator $\mathcal{K}$ is given by [4]

$$\mathcal{K}f(x, y, z, t) = \frac{1}{2\pi}\int_{\mathcal{R}^2}\frac{1}{r}f(x', y', z, \tau)H(\tau)\,dx'\,dy', \tag{5}$$

where $r = \sqrt{(x - x')^2 + (y - y')^2}$, $\tau = t - \frac{r}{c(z)}$ and $H(t)$ is the Heaviside step function.

In the present paper, we use the transverse zeroth and second moments of the fields to reduce the three-dimensional problem to a set of one-dimensional problems. The transverse zeroth and second moments of the field $u(x,y,z,t)$ are defined by

$$u_0(z,t) = \int\int_{\mathcal{R}^2} u(x,y,z,t)\, dx\, dy, \tag{6}$$

$$u_2(z,t) = \int\int_{\mathcal{R}^2} (x^2+y^2) u(x,y,z,t)\, dx\, dy, \tag{7}$$

respectively. Taking zeroth and second moments of Eq. (3), we obtain the split moments

$$\begin{bmatrix} u_0^+ \\ u_0^- \\ u_2^+ \\ u_2^- \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}c(z)\partial_t^{-1} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2}c(z)\partial_t^{-1} & 0 & 0 \\ 0 & -c^3\partial_t^{-3} & \frac{1}{2} & -\frac{1}{2}c(z)\partial_t^{-1} \\ 0 & c^3\partial_t^{-3} & \frac{1}{2} & \frac{1}{2}c(z)\partial_t^{-1} \end{bmatrix} \begin{bmatrix} u_0 \\ u_{0z} \\ u_2 \\ u_{2z} \end{bmatrix} \equiv T \begin{bmatrix} u_0 \\ u_{0z} \\ u_2 \\ u_{2z} \end{bmatrix} \tag{8}$$

(note that $u_0 = u_0^+ + u_0^-$ and $u_2 = u_2^+ + u_2^-$), where the usual notation ∂_t^{-1} for the time integral has be adopted.

The split moments are related to each other by the moments of the reflection operator [5]

$$u_0^-(z,t) = R_0(z,t) * u_0^+(z,t), \tag{9}$$

$$u_2^-(z,t) = R_2(z,t) * u_0^+(z,t) + R_0(z,t) * u_2^+(z,t), \tag{10}$$

where R_0 and R_2 are the zeroth and second moments of the reflection operator, respectively. In the definitions (9) and (10) the following shorthand notation for a time convolution integral has been used

$$f(z,t) * g(z,t) = \int_0^t f(z,t-t')g(z,t')dt'. \tag{11}$$

Note that $R_0(0,t)$ and $R_2(0,t)$ are measurable quantities which later will be used as the input for a simultaneous reconstruction.

The dynamic equation for the split moments is

$$\partial_z \begin{bmatrix} u_0^+ \\ u_0^- \\ u_2^+ \\ u_2^- \end{bmatrix} = \begin{bmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ p_{11} & p_{12} & \alpha & \beta \\ p_{21} & p_{22} & \gamma & \delta \end{bmatrix} \begin{bmatrix} u_0^+ \\ u_0^- \\ u_2^+ \\ u_2^- \end{bmatrix}, \tag{12}$$

where

$$
\begin{cases}
\alpha = -\frac{1}{c}\frac{\partial}{\partial t} - \frac{1}{2}bc + \frac{c_z}{2c}, \\
\beta = -\frac{1}{2}bc - \frac{c_z}{2c}, \\
\gamma = \frac{1}{2}bc - \frac{c_z}{2c}, \\
\delta = \frac{1}{c}\frac{\partial}{\partial t} + \frac{1}{2}bc + \frac{c_z}{2c}, \\
p_{11} = 2c\partial_t^{-1} - bc^3\partial_t^{-2} + 2cc_z\partial_t^{-2}, \\
p_{12} = -bc^3\partial_t^{-2} - 2cc_z\partial_t^{-2}, \\
p_{21} = bc^3\partial_t^{-2} - 2cc_z\partial_t^{-2}, \\
p_{22} = -2c\partial_t^{-1} + bc^3\partial_t^{-2} + 2cc_z\partial_t^{-2},
\end{cases}
\tag{13}
$$

where one sees that the split second moments $u_2^{\pm}$ are not involved in the dynamic equation for split zeroth moments $u_0^{\pm}$ but the split zeroth moments $u_0^{\pm}$ are involved in the dynamic equation for split second moments $u_2^{\pm}$. Consequently, R_2 will not appear in the PDE for R_0 but R_0 will appear in the PDE for R_2.

Using the dynamic equation for the split moments, we obtain the following imbedding equations

$$
R_{0z} = \frac{2}{c}R_{0t} + bcR_0 + \frac{1}{2}(bc + \frac{c_z}{c})R_0 * R_0, \tag{14}
$$

$$
\begin{aligned}
R_{2z} = \; & \frac{2}{c}R_{2t} + (bc^2 - 2c_z)ct + 2c(bc^2t - 2) * R_0 \\
& +(bc^2 + 2c_z)ct * R_0 * R_0 + bcR_2 + (bc + \frac{c_z}{c})R_0 * R_2
\end{aligned}
\tag{15}
$$

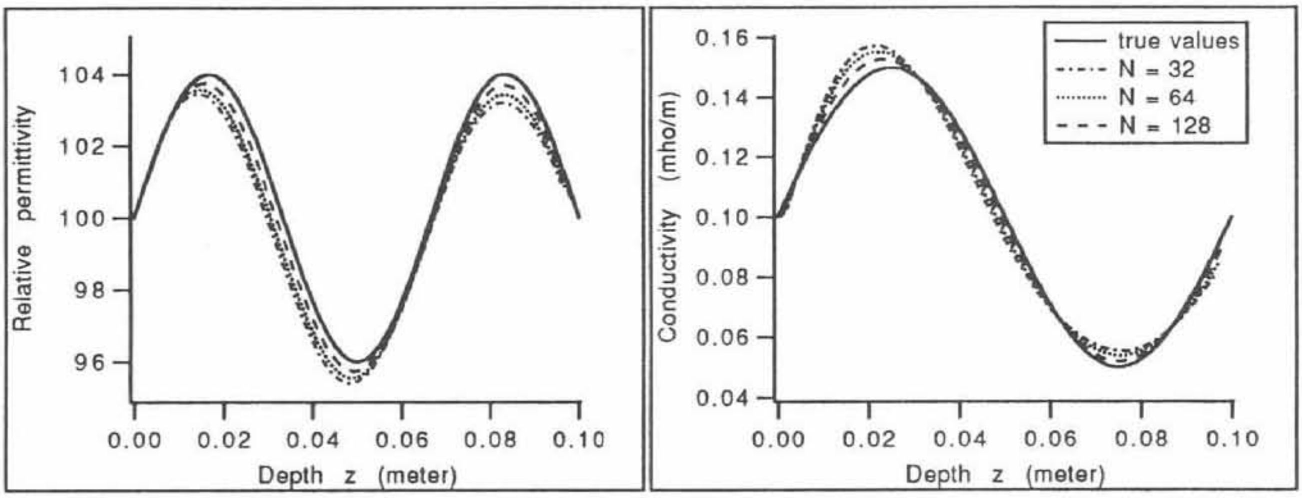
(note that the PDE for R_0 is non-linear in R_0, while the PDE for R_2 is linear in R_2 but coupled with R_0), and the initial conditions

$$
R_0(z,0) = \frac{1}{4}(c_z - bc^2), \tag{16}
$$

$$
\ddot{R}_2(z,0) = \frac{1}{2}c^2(3c_z - 2bc^2), \tag{17}
$$

where $\ddot{R}_2$ denotes the second time derivative of R_2.

In the inverse algorithm, we propagate the measurable boundary values $R_0(0,t)$ and $R_2(0,t)$ to the initial values using the PDEs for $R_0(z,t)$ and $R_2(z,t)$, respectively. The phase velocity $c(z)$ and dissipation coefficient $b(z)$ are then simultaneously reconstructed by the initial conditions (16) and (17). The numerical results for a simultaneous reconstruction using different data points number N are presented in the figures given below. In this numerical example we have considered the case of electromagnetic wave propagation, where $c(z) = 1/\sqrt{\epsilon(z)\mu_0}$, $b(z) = \mu_0\sigma(z)$ (ϵ is the permittivity, σ is the conductivity and μ_0 is the permeability in vacuum).



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