Floquet-Mode Analysis of Pillar-type Photonic Crystal Waveguide Using Spectral-Domain Approach

[#] Yoshimasa NAKATAKE [1] and Koki WATANABE [2]
[1] Graduate School of Engineering, Fukuoka Institute of Technology
3-30-1 Wajiro-Higashi, Higashi-ku, Fukuoka 811-0295, Japan, y-nakatake@m.ieice.org
[2] Department of Information and Communication Engineering,
Fukuoka Institute of Technology
3-30-1 Wajiro-Higashi, Higashi-ku, Fukuoka 811-0295, Japan, koki@fit.ac.jp

1. Introduction

Photonic crystal is a periodic structure consisting of highly contrast dielectrics, in which the electromagnetic wave cannot transmit in a specific wavelength range. It is therefore known that, if localized defects are introduced in the photonic crystal, the electromagnetic fields are strongly confined around the defects. For example, point defects in the photonic crystal work as resonance cavities and line defects work as waveguides. This paper presents a Floquet-mode analysis of the photonic crystal waveguide (PCW) using the spectral-domain approach. For the straight waveguides, the structure maintains the periodicity in the propagation direction, and the Floquet theorem asserts that the electromagnetic fields in the structure can be expressed by superposition of the Floquet-modes [1]. The Floquet-modes of the PCW are obtained by the eigenvalue analysis of the transfer matrix for one periodicity cell in the propagation direction. The periodicity cell that makes up the PCW has imperfect periodicity in the direction perpendicular to wave propagation. Therefore the fields in the structure have continuous spectra. The present analysis uses the pseudo-periodic Fourier transform (PPFT) [2] to consider the discretization scheme in the wavenumber space. The PPFT and its inverse are formally given by

$$\bar{f}(x;\xi) = \sum_{m=-\infty}^{\infty} f(x-md)e^{imd\xi} \tag{1}$$

$$f(x) = \frac{1}{k_d}\int_{-k_d/2}^{k_d/2} \bar{f}(x;\xi)d\xi \tag{2}$$

where d is a positive value usually chosen to be equal with the structural period, ξ is a transform parameter, and $k_d = 2\pi/d$ is the inverse lattice constant.

2. Outline of Formulation

This paper considers the guided Floquet-modes propagating in a PCW schematically shown in Fig. 1. The structure consists of identical circular cylinders that are infinitely long and described by the radius a, the permittivity ε_c, and the permeability μ_c. The cylinders are situated in a surrounding medium with the permittivity ε_s and the permeability μ_s. The cylinder axes parallel to the z-axis are located at $(x, y) = (ld, mh)$ for integer $l \neq 0$ and any integer m. The present paper considers the region $-h/2 \leqq y \leqq h/2$ as a periodicity cell, and analyzes the fields in this region. The fields are supposed to be uniform in the z-direction. Then, the problem becomes two-dimensional, and the fields are decomposed into the transverse magnetic (TM) and the transverse electric (TE) polarizations, in which the magnetic and the electric fields are perpendicular to the

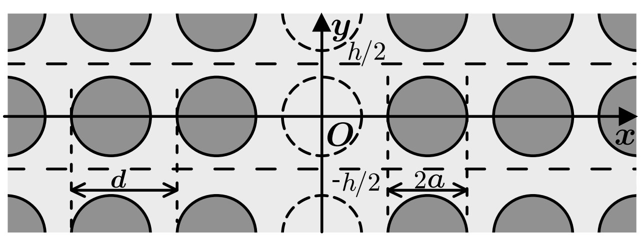

Figure 1: Photonic crystal waveguide

z-axis, respectively. We consider time-harmonic electromagnetic fields assuming a time-dependence in $e^{-i\omega t}$.

The incident field for the periodicity cell consists of the waves propagating in the negative y-direction from the plane $y = h/2$ and the waves propagating in the positive y-direction from the plane $y = -h/2$. Therefore, the incident field transformed by the PPFT $\bar{\psi}^{(i)}(x; \xi, y)$ can be expressed in the plane-wave expansion [2] as

$$\bar{\psi}^{(i)}(x; \xi, y) = \boldsymbol{f}^{(-)}(x, y - h/2; \xi)\bar{\boldsymbol{\psi}}^{(-)}(\xi, h/2) + \boldsymbol{f}^{(+)}(x, y + h/2; \xi)\bar{\boldsymbol{\psi}}^{(+)}(\xi, -h/2) \tag{3}$$

where the column matrices $\boldsymbol{f}^{(\pm)}(x, y; \xi)$ are generated by the plane-waves whose nth-components are given by

$$\left(\boldsymbol{f}^{(\pm)}(x, y; \xi)\right)_n = e^{i(\alpha_n(\xi)\,x \pm \beta_n(\xi)\,y)}, \quad \alpha_n(\xi) = \xi + nk_d, \quad \beta_n(\xi) = \sqrt{{k_s}^2 - \alpha_n(\xi)^2}. \tag{4}$$

$\bar{\boldsymbol{\psi}}^{(\pm)}(\xi, y)$ denote the column matrices of the amplitude corresponding to the plane-waves propagating in the positive and the negative y-direction, respectively, and k_s denotes the wavenumber in the surrounding medium.

The plane-wave is known to be expressed by a superposition of the cylindrical-waves concerning with the Bessel function, and if we choose the reference point of the bases at $(x, y) = (0, 0)$, the incident field is also written in the following form:

$$\bar{\psi}^{(i)}(x; \xi, y) = \boldsymbol{g}^{(J)}(x, y)^t \bar{\boldsymbol{a}}^{(i)}(\xi). \tag{5}$$

The column matrices $\boldsymbol{g}^{(Z)}(x, y)$ gives the cylindrical-wave expansion bases whose nth-components are given by

$$\left(\boldsymbol{g}^{(Z)}(x, y)\right)_n = Z_n(k_s\rho(x, y))e^{in\phi(x,y)}, \quad \rho(x, y) = \sqrt{x^2 + y^2}, \quad \phi(x, y) = \arg(x + iy) \tag{6}$$

where Z specifies the cylinder functions associating to the cylindrical-wave bases in such a way that $Z = J$ denotes the Bessel function and $Z = H^{(1)}$ denotes the Hankel function of the first kind. The coefficient matrix $\bar{\boldsymbol{a}}^{(i)}(\xi)$ is derived as

$$\bar{\boldsymbol{a}}^{(i)}(\xi) = \boldsymbol{A}^{(-)}(\xi)^t \boldsymbol{F}(h/2; \xi)^t \bar{\boldsymbol{\psi}}^{(-)}(\xi, h/2) + \boldsymbol{A}^{(+)}(\xi)^t \boldsymbol{F}(h/2; \xi)^t \bar{\boldsymbol{\psi}}^{(+)}(\xi, -h/2) \tag{7}$$

$$(\boldsymbol{F}(y; \xi))_{n,m} = \delta_{n,m} e^{i\beta_n(\xi)y}, \quad \left(\boldsymbol{A}^{(\pm)}(\xi)\right)_{n,m} = \left(\frac{i\alpha_n(\xi) \pm \beta_n(\xi)}{k_s}\right)^m \tag{8}$$

for the Kronecker delta $\delta_{n,m}$. Applying the inverse PPFT (2) to Eq.(5), and choosing the reference points of the bases at $(x, y) = (nd, 0)$ for an integer n, we may obtain the cylindrical-wave expansion expression of the original incident field $\psi^{(i)}(x, y)$ as

$$\psi^{(i)}(x, y) = \boldsymbol{g}^{(J)}(x - nd, y)^t \boldsymbol{a}_n^{(i)}, \quad \boldsymbol{a}_n^{(i)} = \frac{1}{k_d}\int_{-k_d/2}^{k_d/2} \bar{\boldsymbol{a}}^{(i)}(\xi)e^{ind\xi}d\xi. \tag{9}$$

On the other hand, the scattered field $\psi^{(s)}(x, y)$ consists of the outward propagating waves from the cylinders, and the scattered field outside the cylinder is given in the following form:

$$\psi^{(s)}(x, y) = \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \boldsymbol{g}^{(H^{(1)})}(x - ld, y)^t \boldsymbol{a}_l^{(s)} \tag{10}$$

where $\boldsymbol{a}_l^{(s)}$ denotes the column matrix generated by the expansion coefficients of the scattered waves from the lth-cylinder. Applying the PPFT to the scattered field, the transformed field are expressed as follows:

$$\bar{\psi}^{(s)}(x; \xi, y) = \sum_{l=-\infty}^{\infty} \boldsymbol{g}^{(H^{(1)})}(x - ld, y)^t \bar{\boldsymbol{a}}^{(s)}(\xi)e^{ild\xi} \tag{11}$$

$$\bar{\boldsymbol{a}}^{(s)}(\xi) = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \boldsymbol{a}_m^{(s)} e^{imd\xi}. \tag{12}$$

The scattered field transformed by the PPFT is rewritten in the plane-wave expansion as

$$\bar{\psi}^{(s)}(x;\xi,y) = \begin{cases} \boldsymbol{f}^{(+)}(x,y;\xi)^t \boldsymbol{B}^{(+)}(\xi)^t \bar{\boldsymbol{a}}^{(s)}(\xi) & \text{for } y \geqq 0 \\ \boldsymbol{f}^{(-)}(x,y;\xi)^t \boldsymbol{B}^{(-)}(\xi)^t \bar{\boldsymbol{a}}^{(s)}(\xi) & \text{for } y < 0 \end{cases} \tag{13}$$

$$\left(\boldsymbol{B}^{(\pm)}(\xi)\right)_{n,m} = \frac{2}{d\beta_m(\xi)} \left(\frac{-i\alpha_m(\xi) \pm \beta_m(\xi)}{k_s})\right)^n . \tag{14}$$

The coefficient matrices of the incident and the scattered fields for each cylinder are known to be related by the transition-matrix (T-matrix), and the recursive transition-matrix algorithm (RTMA) [3] yields the following relation:

$$\boldsymbol{a}_m^{(s)} = \boldsymbol{T} \left(\boldsymbol{a}_m^{(i)} + \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \boldsymbol{G}^{(H^{(1)})}((m-l)d,0)^t \boldsymbol{a}_l^{(s)} \right) \tag{15}$$

with

$$\left(\boldsymbol{G}^{(Z)}(x,y)\right)_{n,m} = Z_{n-m}(k_s\rho(x,y))e^{i(n-m)\phi(x,y)} \tag{16}$$

$$(\boldsymbol{T})_{n,m} = \delta_{n,m} \begin{cases} \frac{\zeta_s J_n(k_sa)J_n'(k_ca) - \zeta_c J_n'(k_sa)J_n(k_ca)}{\zeta_c H_n^{(1)\prime}(k_sa)J_n(k_ca) - \zeta_s H_n^{(1)}(k_sa)J_n'(k_ca)} & \text{for TM-polarization} \\ \frac{\zeta_c J_n(k_sa)J_n'(k_ca) - \zeta_s J_n'(k_sa)J_n(k_ca)}{\zeta_s H_n^{(1)\prime}(k_sa)J_n(k_ca) - \zeta_c H_n^{(1)}(k_sa)J_n'(k_ca)} & \text{for TE-polarization} \end{cases} \tag{17}$$

where ζ_c and ζ_s denote the characteristic impedances of the cylinder and the surrounding medium respectively, and k_c is the wavenumber inside the cylinder.

Substituting Eq. (15) into Eq. (12), we obtain the relationship between the coefficient matrices of the incident and scattered fields transformed by the PPFT as

$$\left(\boldsymbol{T}^{-1} - \boldsymbol{L}(\xi)\right)\bar{\boldsymbol{a}}^{(s)}(\xi) + \frac{1}{k_d}\int_{-k_d/2}^{k_d/2} \boldsymbol{L}(\xi')\bar{\boldsymbol{a}}^{(s)}(\xi')d\xi' = \bar{\boldsymbol{a}}^{(i)}(\xi_l) - \frac{1}{k_d}\int_{-k_d/2}^{k_d/2} \bar{\boldsymbol{a}}^{(i)}(\xi')d\xi' \tag{18}$$

with

$$\boldsymbol{L}(\xi) = \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \boldsymbol{G}^{(H^{(1)})}(-ld,0)^t e^{ild\xi}. \tag{19}$$

To solve the integral equation (18), we introduce a discretization in the transform parameter ξ. We take L sample points $\{\xi_l\}_{l=1}^{L}$, therefore Eq. (18) is properly satisfied at these points. Then we have

$$\left(\boldsymbol{T}^{-1} - \boldsymbol{L}(\xi_l)\right)\bar{\boldsymbol{a}}^{(s)}(\xi_l) + \frac{1}{k_d}\sum_{l'=1}^{L} w_{l'}\boldsymbol{L}(\xi_{l'})\bar{\boldsymbol{a}}^{(s)}(\xi_{l'}) = \bar{\boldsymbol{a}}^{(i)}(\xi_l) + \frac{1}{k_d}\sum_{l'=1}^{L} w_{l'}\bar{\boldsymbol{a}}^{(i)}(\xi_{l'}) \tag{20}$$

where $\{w_l\}_{l=1}^{L}$ denote the weight factors determined by the appropriate numerical integration scheme. The coefficient matrices of the scattered field at the sample point is obtained by

$$\widetilde{\boldsymbol{a}}^{(s)} = \widetilde{\boldsymbol{M}}^{-1}\widetilde{\boldsymbol{C}}\widetilde{\boldsymbol{a}}^{(i)} \tag{21}$$

with

$$\widetilde{\boldsymbol{a}}^{(i)} = \begin{pmatrix} \bar{\boldsymbol{a}}^{(i)}(\xi_1) \\ \vdots \\ \bar{\boldsymbol{a}}^{(i)}(\xi_L) \end{pmatrix}, \quad \widetilde{\boldsymbol{a}}^{(s)} = \begin{pmatrix} \bar{\boldsymbol{a}}^{(s)}(\xi_1) \\ \vdots \\ \bar{\boldsymbol{a}}^{(s)}(\xi_L) \end{pmatrix}, \quad \widetilde{\boldsymbol{M}} = \begin{pmatrix} \boldsymbol{M}_{1,1} & \cdots & \boldsymbol{M}_{1,L} \\ \vdots & \ddots & \vdots \\ \boldsymbol{M}_{L,1} & \cdots & \boldsymbol{M}_{L,L} \end{pmatrix}, \quad \widetilde{\boldsymbol{C}} = \begin{pmatrix} \boldsymbol{C}_{1,1} & \cdots & \boldsymbol{C}_{1,L} \\ \vdots & \ddots & \vdots \\ \boldsymbol{C}_{L,1} & \cdots & \boldsymbol{C}_{L,L} \end{pmatrix} \tag{22}$$

$$\boldsymbol{C}_{l,l'} = \left(\delta_{l,l'} + \frac{w_{l'}}{k_d}\right)\boldsymbol{I}, \quad \boldsymbol{M}_{l,l'} = \delta_{l,l'}\left(\boldsymbol{T}^{-1} - \boldsymbol{L}(\xi_l)\right) + \frac{w_{l'}}{k_d}\boldsymbol{L}(\xi_{l'}) \tag{23}$$

where $\boldsymbol{I}$ is the identity matrix. We can obtain a relation between the amplitudes of plane-waves $\widetilde{\boldsymbol{\psi}}^{(\pm)}(-h/2)$ and $\widetilde{\boldsymbol{\psi}}^{(\pm)}(h/2)$ as

$$\begin{pmatrix} \widetilde{\boldsymbol{\psi}}^{(+)}(h/2) \\ \widetilde{\boldsymbol{\psi}}^{(-)}(h/2) \end{pmatrix} = \widetilde{\boldsymbol{F}} \begin{pmatrix} \widetilde{\boldsymbol{\psi}}^{(+)}(-h/2) \\ \widetilde{\boldsymbol{\psi}}^{(-)}(-h/2) \end{pmatrix} \tag{24}$$

with

$$\widetilde{\boldsymbol{F}} = \begin{pmatrix} \boldsymbol{S}_{12} - \boldsymbol{S}_{11}\,\boldsymbol{S}_{21}^{-1}\,\boldsymbol{S}_{22} & \boldsymbol{S}_{11}\,\boldsymbol{S}_{21}^{-1} \\ -\boldsymbol{S}_{21}^{-1}\,\boldsymbol{S}_{22} & \boldsymbol{S}_{21}^{-1} \end{pmatrix} \tag{25}$$

$$\boldsymbol{S}_{11} = \widetilde{\boldsymbol{B}}^{(+)t}\widetilde{\boldsymbol{M}}^{-1}\widetilde{\boldsymbol{C}}\widetilde{\boldsymbol{A}}^{(-)t}, \qquad \boldsymbol{S}_{12} = \widetilde{\boldsymbol{B}}^{(+)t}\widetilde{\boldsymbol{M}}^{-1}\widetilde{\boldsymbol{C}}\widetilde{\boldsymbol{A}}^{(+)t} + \widetilde{\boldsymbol{V}} \tag{26}$$

$$\boldsymbol{S}_{21} = \widetilde{\boldsymbol{B}}^{(-)t}\widetilde{\boldsymbol{M}}^{-1}\widetilde{\boldsymbol{C}}\widetilde{\boldsymbol{A}}^{(-)t} + \widetilde{\boldsymbol{V}}, \qquad \boldsymbol{S}_{22} = \widetilde{\boldsymbol{B}}^{(-)t}\widetilde{\boldsymbol{M}}^{-1}\widetilde{\boldsymbol{C}}\widetilde{\boldsymbol{A}}^{(+)t} \tag{27}$$

$$\widetilde{\boldsymbol{A}}^{(\pm)} = \begin{pmatrix} \boldsymbol{A}^{(\pm)}(\xi_1)^t\boldsymbol{F}(h/2;\xi_1)^t & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \boldsymbol{A}^{(\pm)}(\xi_L)^t\boldsymbol{F}(h/2;\xi_L)^t \end{pmatrix} \tag{28}$$

$$\widetilde{\boldsymbol{B}}^{(\pm)} = \begin{pmatrix} \boldsymbol{F}(h/2;\xi_1)\boldsymbol{B}^{(\pm)}(\xi_1)^t & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \boldsymbol{F}(h/2;\xi_L)\boldsymbol{B}^{(\pm)}(\xi_L)^t \end{pmatrix}, \qquad \widetilde{\boldsymbol{V}} = \begin{pmatrix} \boldsymbol{F}(h;\xi_1) \\ \vdots \\ \boldsymbol{F}(h;\xi_L) \end{pmatrix}. \tag{29}$$

The Floquet modes of PCW are calculated by the eigenvalue analysis of the transfer matrix $\widetilde{\boldsymbol{F}}$ given by Eq. (25) [4]. The propagation constants of the Floquet-modes are given as

$$\eta_n = -i\frac{\mathrm{Ln}(\gamma_n)}{h} \tag{30}$$

for $n = 1, \dots, 2L(2N+1)$ (N is the truncation order for plane-wave expansions), where Ln is the principal natural logarithm function, and γ_n is the nth-eigenvalues of $\widetilde{\boldsymbol{F}}$.

3. Conclusion

This paper has provided the spectral-domain formulation of the pillar-type PCW based on the RTMA with the use of the PPFT. The periodicity cell that makes up the PCW has imperfect periodicity in the direction perpendicular to wave propagation. The field transformed by PPFT has a periodic property in terms of the transform parameter. To solve the integral equation, we introduced a discretization in the transform parameter. Then the Floquet-modes can be obtained by the eigenvalue calculations of the transfer matrix without using periodic boundary conditions. The present formulation can calculate not only the guided-modes but also the evanescent-modes.

References

[1] J. D. Joannopoulos, R. D. Meade, and J. N. Winn, *Photonic Crystals: Molding the Flow of Light*, Princeton Univ. Press, Princeton, 1995.

[2] K. Watanabe and K. Yasumoto, "Two-Dimensional Electromagnetic Scattering of Non-Plane Incident Waves by Periodic Structures," Progress In Electromagnetics Research, Vol. 74, pp. 241–271, 2007.

[3] W. C. Chew, *Waves and Fields in Inhomogeneous Media*, Van Nostrand Reinhold, New York, 1990.

[4] Y. Nakatake and K. Watanabe, "Numerical Analysis of Two-Dimensional Photonic Crystal Waveguide Devices Using Periodic Boundary Conditions," IEICE Trans. Electron., Vol. E94-C, No. 1, pp. 32–38, 2011.