ON STABILITY OF 2D AND 3D SOLITONS IN PLASMA
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The propagation of nonlinear waves and solitons (for example, ion-acoustic and magneto-sonic waves) in plasma is described by equation [1]

$$
\begin{equation*}
\partial_{t} u=\partial_{x}(\delta \mathscr{H} / \delta u) \tag{1}
\end{equation*}
$$

with Hamiltonian

$$
\begin{equation*}
\mathscr{H}=\int\left[-\frac{\varepsilon}{2}\left(\partial_{x} u\right)^{2}+\frac{\lambda}{2}\left(\partial_{x}^{2} u\right)^{2}+\frac{1}{2}\left(\nabla_{\perp} \partial_{x} v\right)^{2}-u^{3}\right] d \vec{r} \tag{2}
\end{equation*}
$$

where $\lambda= \pm 1, \quad \partial_{x}^{2} v=u$, and $\varepsilon$ is obtained from dispersion law for acoustic waves in plasma (see, for example, [2]). Second term in expression (2) plays a dominant role when $|\mathcal{E}| \rightarrow 0$ and, moreover, their significance is principal when the solitons dynamics is considerated, i.e. when nonlinear and dispersive (being proportional to $\varepsilon$ ) terms counterbalance one another [3,4] . At this, the solitons stability is a fundamental property of these structures in nonlinear waves physics.

This paper presents the analysis of stability of $2 \mathrm{D}\left(\partial_{\mathrm{z}}=\right.$ $=0$ ) and 3D ( $\partial_{y z} \neq 0$ ) solitons propagating in plasma and other dispersive media which are described by eq. (1) for $\lambda= \pm 1$, $\varepsilon \gtrless 0$.

The stationary solutions of eq. (1) are defined from variation equation

$$
\begin{equation*}
\delta^{2}\left(\mathscr{H}+v P_{x}\right)=0, \quad P_{x}=\frac{1}{2} \int u^{2} d \vec{r} \tag{3}
\end{equation*}
$$

where $v$ has a sense of Lagrange's factor. Eq. (3) illustrates the fact that all finite solutions of eq. (1) are the stationary points of Hamiltonian (2) for fixed $P_{x}$.

Let us consider the problem of stability. In conformity with Lyapunov's theorem, in the dynamic system the stationary points which answer maximum or minimum of $\mathscr{H}_{b}$ are absolutely stable。The locally stable solutions take a place if this extremum is local. Thus, it is needed to prove that $\mathscr{H}$ is bounded (from below) for fixed $P_{x}$.

Let us consider in real vector space $R$ the scale transformations

$$
\begin{equation*}
u\left(x, \vec{r}_{\perp}\right) \rightarrow \zeta^{-1 / 2} \eta^{(1-\alpha) / 2} u\left(x / \zeta, \vec{r}_{\perp} / \eta\right) \tag{4}
\end{equation*}
$$

(where d is the problem dimension and $\zeta, \eta^{\perp} \in R$ ) conserving $\mathrm{P}_{\mathbf{x}}$. Hamiltonian assumes a form
$\mathscr{H}(\zeta, \eta)=a \zeta^{-2}+b \zeta^{2} \eta^{-2}-c \zeta^{-1 / 2} \eta^{(1-d) / 2}+e \zeta^{-4}$
where $a=-(\varepsilon / 2) \int\left(\partial_{x} u\right)^{2} d \vec{r}, b=(1 / 2) \int\left(\nabla_{\perp} \partial_{x} v\right)^{2} d \vec{r}, c=\int u^{3} d \vec{r}, e=$ $=(\lambda / 2) \int\left(\partial \frac{2}{x} u\right)^{2} d \vec{r}$. The necessary conditions of extremum existence are

$$
\begin{equation*}
\partial_{\zeta} \mathscr{H}=0, \quad \partial_{2} \mathscr{H}=0, \tag{6}
\end{equation*}
$$

and we can obtain the extremum coordinates ( $\zeta_{i}, \eta_{j}$ ) from
ones. The conditions

$$
\left|\begin{array}{ll}
\partial_{\zeta}^{2} \mathscr{H}\left(\zeta_{i}, \eta_{j}\right) & \partial_{\zeta_{\eta}}^{2} \mathscr{H}\left(\zeta_{i}, \eta_{j}\right) \\
\partial_{\eta}^{2} \mathscr{H}_{B}\left(\zeta_{i}, \eta_{j}\right) & \partial_{\eta}^{2} \mathscr{H}\left(\zeta_{i}, \eta_{j}\right)
\end{array}\right|>0,
$$

are the sufficient conditions of $\mathscr{H}$ (local) minimum existence.

Let us consider $2 D$ case ( $\mathrm{d}=2$ ). At this, eqs. (6) are the system [1]

$$
\begin{align*}
& G \equiv\left(c^{4} / 32 b\right) t^{4}-(a t+2 e)^{3}=0  \tag{8}\\
& t=\zeta^{2}, \quad \eta=\left[(4 b / c)^{2} \zeta^{5}\right]^{1 / 3}
\end{align*}
$$

An analysis of eq. (8) (see ref. [1]) shows that it has one positive root $t \in R$ for each four of values of $a, b, c$, $e \in R$ for $e>0, a \geqslant 0$, and two positive ones $t_{1,2} \in R$ for $e<0, a>0$, and in case of $e<0, a \leqslant 0$ $t \notin R$.

Ineqs. (7) for $d=2$ with a due account of expressions (8) are led to the form

$$
\begin{align*}
& G-\left(C_{11} a^{3} t^{3}+C_{12} a^{2} e t^{2}+C_{13} a e^{2} t+C_{14} e^{3}\right)<0,  \tag{9.1}\\
& G-\left(C_{21} a^{3} t^{3}+C_{22} a^{2} e t^{2}+C_{23} a e^{2} t+C_{24} e^{3}\right)<0 \tag{9.2}
\end{align*}
$$

where $C_{n m}>0$ are constants. It follows that the conditions (7) are carried out on the set $S_{t} \subset R$ of solutions of system (8) for $e>0$, $a \geqslant 0$, and, consequently, the Hamiltonian is bounded from below. Solving ineqso. (9) in $R$ space for $e>0$, $a<0$ we obtain that $\sup _{(9.1}=\left(3 C_{11}\right)^{-1}\left[2 C_{1} \cos \left(\varphi_{1} / 3\right)-C_{12}\right] a^{-1}$ e inf $A_{t}=0$ for $S_{t}{ }^{(8.1)} \cap S_{t}{ }^{(9.2)}=A_{t} C_{R}$. Here $C_{1}=\left(C_{\left.12-3 C_{11} C_{13}\right)}{ }^{1 / 2}\right.$, $\varphi_{1}=$ Arccos $\left\{\left(2 C_{1}^{3}\right)^{-1}\left[C_{12}\left(C_{12}^{2}-3 C_{1}^{2}\right)-27 C_{1}^{2} C_{14}\right]\right\}$ Accounting the
results of rei. [1] and noting that $S_{t} \cap A_{t} \not \equiv \varnothing$ we obtain that for e >0, a $<0$ the sufficient condition of the $\mathscr{H}_{\mathfrak{G}}$ local minimum existence is $S_{t} \subseteq A_{t}$, i。e.

$$
\begin{equation*}
(a / c)(b / e)^{1 / 4} \geqslant\left(6 c_{11}\right)^{-1}\left[c_{1} \cos \left(\varphi_{1} / 3\right)-c_{12} / 2\right] . \tag{10}
\end{equation*}
$$

Considering by analogy ineqs. (9) for e $\leq 0$, a $>0$ we obtain that inf $\mathrm{B}_{t^{(1)}}=\left(3 \mathrm{C}_{21}\right)^{-1}\left[2 \mathrm{C}_{2} \mathrm{ch}\left(\varphi_{2} / 3\right)-\mathrm{C}_{22}\right] \mathrm{a}_{(2)}^{-1} e^{2}$ sup $\mathrm{B}_{1}^{(1)}=$ $=\left(3 C_{11}\right)^{-1}\left[2 C_{1} \cos \left(\varphi_{1} / 3+4 \pi / 3\right)-C_{12}\right] a^{-1} e^{2} \inf ^{2} \mathrm{~B}_{\mathrm{t}}^{(2)}=\left(3 C_{11}\right)^{-1}\left[C_{2} C_{1}\right.$
 $\left.-3 C_{2}^{2}\right)-27 C_{2}^{2} 1_{24} \mathcal{B}_{t}{ }^{(1)} \subset S_{t} \Rightarrow B_{t}{ }^{(1)} \cap S_{t}=B_{t}(1)$ and $B_{t}{ }^{(2)} \cap S_{t}=\varnothing$ we tain that $B_{t_{4}} \subset S_{t} \Rightarrow B_{t_{4}}{ }^{(1)} \cap S_{t}=B_{t}{ }^{(1)}$ and $B_{t}{ }^{(2)} \cap S_{t}=\varnothing$. Assuming that $a^{4} b / c^{4} e \leqslant-2^{4} 3^{-3} \cdot Q^{-1}$ with $Q>1$ we obtain with a due account of results of ref. [1] the value of $Q=-2^{8} 3^{-3}$ (T+ $+2) / T^{2}$ with $T=$ inf $B_{t}^{(1)} \cdot a e^{-1}$ which answers the sufficient condition $^{\prime}$ of the $\mathscr{H}$ local minimum condition, namely: infs $_{t}=$ $\inf B_{t}^{(1)}$, i.e.

$$
\begin{equation*}
a^{4} b / c^{4} e \leqslant 2^{-4} T^{2} /(T+2) \tag{11}
\end{equation*}
$$

Let us consider now 3D case ( $d=3$ ). At this, for each four of values of $a, b, c, e \in R$ with $a \neq 0$ we obtain (see also
[1]):

$$
\begin{align*}
\zeta_{i} & =(16 a b)^{-1}\left(3 c^{2} \pm \sqrt{9 c^{4}-512 a b^{2} e}\right), \\
\gamma_{j} & =(2 b / c) \zeta_{i}^{5 / 2} ; \quad i=1,2 ; \quad j=1,2,3,4 \tag{12}
\end{align*}
$$

Let us note that ( $\zeta_{i}, \eta_{j}$ ) $\notin R$ for $\zeta_{i}<0$, and, therefore, further we'll consider only the rocts $\xi_{i}>0$ (we rule out equality $\zeta_{i}=0$ with a due account of $\mathrm{e} \neq \hat{0}$, otherwise eq. (1) degenerates to usual the Kadomtsev-Petviashvili equation). Ineqs. (7) with a due account of expressions (12) take the form

$$
\begin{align*}
& a \zeta^{2}-\left(c^{2} / 2 b\right) \zeta+10 e / 3>0  \tag{13.1}\\
& a \zeta^{2}+\left(c^{2} / 48 b\right) \zeta+10 e / 3>0 \tag{13.2}
\end{align*}
$$

In case of $e>0, a>0$ condition $\zeta_{i} \in R$, i.e.

$$
\begin{equation*}
\mathrm{c}^{4} \geqslant(512 / 9) \mathrm{ab}^{2} \mathrm{e}, \tag{14}
\end{equation*}
$$

brings $\zeta_{1},>0$ in its train. Elementary analysis shows that $S{ }_{S 1}^{(12)} n^{1}, S_{S}^{(13,1)}=\varnothing$ and, if the strict inequality takes a
 the $\mathscr{H}_{b}$ local minimum for $e^{s_{2}}>0$, $a>0$, it's enough that

$$
\begin{equation*}
a b^{2} e / c^{4}<9 / 512 . \tag{15}
\end{equation*}
$$


 $S_{S 2}^{(12)} \subset S_{S_{2}}^{(13)}$. Thus, function $\mathscr{H}(\zeta, \eta)$ is bounded from below for any e \gg0, a < 0 .

The analogous consideration for $e<0$ shows that $\zeta_{1,2}<$ $<0$ for each four of values of $a, b, c, e \in R$ if $a<0$, $\quad(12)$, $n$
and ineq. (14) is carried out. Consequently we have

 /c) $\xi_{i}^{5 / 2}, i=1, j=1,2$ instead of eqs. (12) for each three of
 $<0$, and $S_{\zeta} \subset S_{\zeta}{ }^{(13)}$ for e $>0$.

Summing the results obtained above let's conclude the following. In 2D case for fixed $P_{x}$ the Hamiltonian of system (1), (2) is bounded from below for the eq. (1) coefficients $\lambda=1$, $\varepsilon \leqslant 0$, and, consequently, the $2 D$ solitons are absolutely stable in this case. For $\lambda=1, \varepsilon>0$ and $\lambda=-1, \varepsilon<0$ the Hemiltonian has the local minima if, accordingly, conditions (10) and (11) are carried out. The 2D solitons are local stable in these cases. In 3D case for fixed $P_{X}$ the Hamiltonian has the local minimum for $\lambda=1, \varepsilon \leqslant 0$ if condition (15) is carried out, and it is bounded from below for $\lambda=1, \varepsilon>0$. The 3D solitons are, accordingly, local and absolutely stable in these cases.

The analysis of $\mathscr{H}$ bounding for the solutions of eq.(1) obtained numerically in ref. [3,4] for $\mathrm{d}=2,3$ enabled us to corroborate the results of numerical simulation of the solitons dynamics. The application of this analysis to the problem of the fast magneto-sonic waves beam's propagation in magnetized plasma enabled us to prove [2], for example, that the three-dimensional beam propagating at $\theta$ angle to magnetic field doesn't focuse and becomes stationary and stable in the
cone of
when inequality

$$
\theta<\operatorname{arctg}(\mathbb{M} / m)^{1 / 2}
$$

$$
\left(\frac{m}{M}-\operatorname{ctg}^{2} \theta\right)^{2}\left[\operatorname{ctg}^{4} \theta\left(1+\operatorname{ctg}^{2} \theta\right)\right]^{-1}>4 / 3
$$

i.e. when e $>0$, a<0 in expression (2). Let us note also that obtained here results give us the possibility to interpret correctly our numerical and theoretical results on dynamics of the internal gravity waves solitons induced by sources of pulse-type which propagate at heights of ionosphere F region [5,6] from the point of view of such solitons stability。

## References

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