THE DISPERSION CHARACTERISTICS OF THE FD-TD METHOD

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Introduction

The Finite Difference Time Domain (FD-TD) method[1] based on Yee's algorithm has been recently used to solve the problems of waveguiding structures such as cut-off frequencies of finlines[1] or scatterring parameters of microstrip discontinuities[2]-[3]. Even though the FD-TD technique has been explored quite successfully in the past(see for example [4]), the numerical mesh network of the algorithm causes numerical error in the phase and group velocities of the propagating wave, primarily due to its dispersive nature[5].

In order to assess the numerical error caused by the dispersion for solutions of a TEM wave or waveguide modes, we present an explicit expression of the phase and group velocities of the three-dimensional FD-TD algorithm derived from the following dispersion relation[4].

$$sin^2(\omega\Delta t/2) \;=\; s^2[sin^2(k_x\Delta x/2) + sin^2(k_y\Delta y/2) + sin^2(k_z\Delta z/2)] \quad (1)$$

Formulation

For a wave whose angles of the propagating direction is given by α, β and γ from x, y, and z coordinates respectively ($k_x = kcos\alpha, k_y = kcos\beta, k_z = kcos\gamma$ and $k\Delta x = k\Delta y = k\Delta z = k\Delta l = \theta$), it can be shown that the group and phase velocities in free space are given

$$v_g \;=\; s\sqrt{sin^2(\theta cos\alpha) + sin^2(\theta cos\beta) + sin^2(\theta cos\gamma)}/sin(\omega\Delta t) \quad (2)$$

$$v_p \;=\; \frac{2sin^{-1}[s\sqrt{sin^2(\theta cos\alpha/2) + sin^2(\theta cos\beta/2) + sin^2(\theta cos\gamma/2)}]}{s\theta} \quad (3)$$

where s is the stability factor given as $s = c\Delta t/\Delta l$. Above expressions are mixed forms which show the seperate effect of ω and k for the propagating wave. Depending upon application, it may be sometimes convenient to express them in terms of only ω or k. However, this is possible only for the

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cases where propagations occur to the axial direction(A), to the diagonal direction(B) of a plane, and to the diagonal direction(C) of a cubic cell in free space(see figure 1. b). For other directional waves, both the dispersion relation (1) and the formula of (2) or (3) are required. The phase and group velocities of these special cases are:

$$v_g = \sqrt{1-P^2}/cos(\omega\Delta t/2)\cdots direction\cdots A \quad (4)$$
$$= \sqrt{1-P^2/2}/cos(\omega\Delta t/2)\cdots direction\cdots B \quad (5)$$
$$= \sqrt{1-P^2/3}/cos(\omega\Delta t/2)\cdots direction\cdots C \quad (6)$$

and

$$v_p = \frac{\pi\Delta t/\lambda}{sin^{-1}(P)}\cdots direction\cdots A \quad (7)$$
$$= \frac{\pi\Delta t/\lambda}{\sqrt{2}sin^{-1}(P/\sqrt{2})}\cdots direction\cdots B \quad (8)$$
$$= \frac{\pi\Delta t/\lambda}{\sqrt{3}sin^{-1}(P/\sqrt{3})}\cdots direction\cdots C \quad (9)$$

where $P = sin(s\pi\Delta t/\lambda)/s$ and all velocities are normalized by the velocity of light.

It is found that the group and phase velocities of the wave to the diagonal direction of a cubic cell(direction C) show no dispersion(figure 2) and that except for those of the specific directions(A, B and C) the group velocities in other directions do not decrease to zero but have finite cut-off frequencies as shown in figure 3. This holds only with a maximun value of the stability factor: $s = 1/\sqrt{3}$.

For the cylindrical modal solutions, the phase and group velocities of a simple waveguide with a rectangular cross-section which supports the TE or TM modes were obtained with the procedure similar to the previous cases. They are :

$$v_g = \frac{ssin(2sin^{-1}\sqrt{P^2-Q^2})}{sin(2s\pi\Delta l/\lambda)} \quad (10)$$
$$v_p = \frac{\pi\Delta l/\lambda}{sin^{-1}\sqrt{P^2-Q^2}} \quad (11)$$

then,

$$v_g v_p = \frac{\omega\Delta tsin(2sin^{-1}\sqrt{P^2-Q^2})/2}{sin^{-1}(\sqrt{P^2-Q^2})sin(\omega\Delta t)} \quad (12)$$

where Q is given by $Q = sin(\pi \Delta l/\lambda_c)$ andλ_c is the cutoff wavelengh. These are compared with the theoretical results in figure 4. This formula also holds for other types of cylindrical waveguides with different cross-sections provided that the cut-off frequency is properly replaced by the corresponding value. The group velocities always cross over at a specific frequency. At this frequency, the numerical group velocity coincides with the theoretical group velocity. It is found that this relative coss-over frequecy (λ_c/λ) depends on the stability factor s and have slight dependancy on the mesh size of dl. However, the numerical phase velocity is always less than the theoretical velocity and minimum error occurs at certain frequency which is different from the cross-over frequency of the group velocity.

conclusion

We derived the numerical phase and group velocity characteristics of the FD-TD algorithm from the dispersion relation for wave propagation in TEM, TE or TM modes. From these dispersion characteristics, numerical errors due to dispersion can be easily assessed and would be used in determining the mesh size and the stability factor s to minimize numerical errors.

References

1. D.H. Choi and W.J.R. Hoefer,"The Finite Difference Time Domain Method and Its Application to Eigenvalue Problems", IEEE Trans., Microwave Theory Tech., vol MTT-34, pp.1464-1470, Dec. 1986.

2. X. Zhang, J. Fang, K.K. Mei and Y. Liu,"Calculation of the Dispersive Characteristics of Microstrips by the Time-Domain Finite Difference Method", IEEE Trans. Microwave Theory Tech. vol. MTT-36, No.2, pp 263-267, Feb. 1988.

3. X. Zhang and K.K. Mei,"Time Domain Finite Difference Approach for the Calculation of Microstrip Open-Circuit End Effect", IEEE MTT-S International Microwave Symposium Digest pp 363-366, June 1988.

4. A. Taflove and A. R. Umashankar,"Finite Difference Time Domain Modeling of Electromagnetic Wave Scattering and Interaction Problems", IEEE Antennas and Propagation Society Newsletter, pp 5-20 April 1988.

5. L. N. Trefethen,"Group velocity in Finite Difference Schemes", Siam Review vol. 24, No.2, pp 113-135, April 1982.

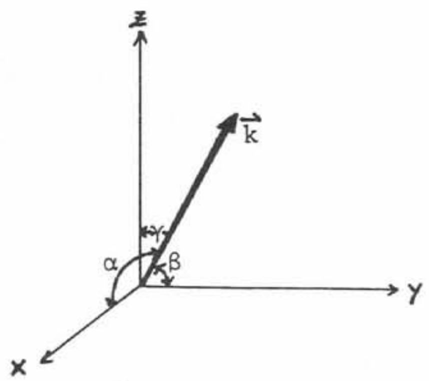

Fig. 1 a) Wave Propagating Angles with respect to the coordinates.

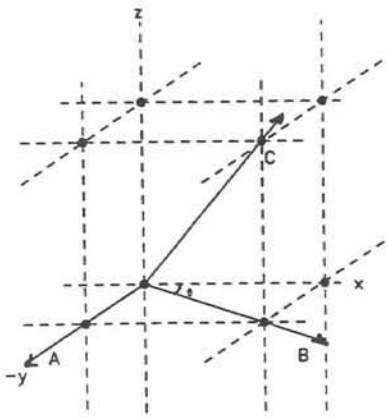

Fig. 1 b) Three Fundamental Directions of TEM Wave Propagation

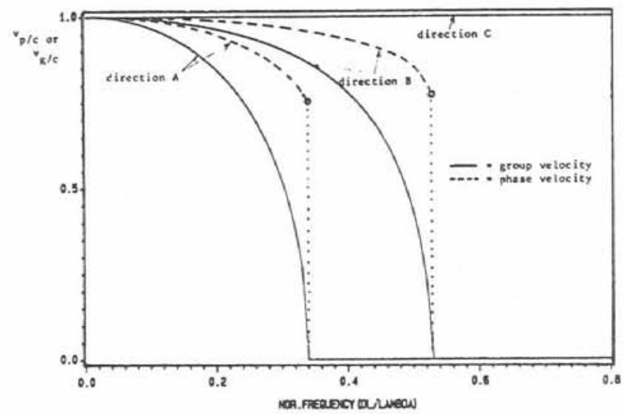

Fig. 2 Group and Phase Velocities of a TEM Wave Propagating to directions, A,B,C.

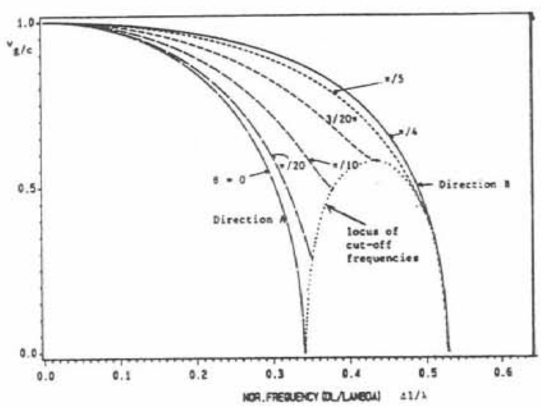

Fig. 3 Cut off Frequencies for different angle

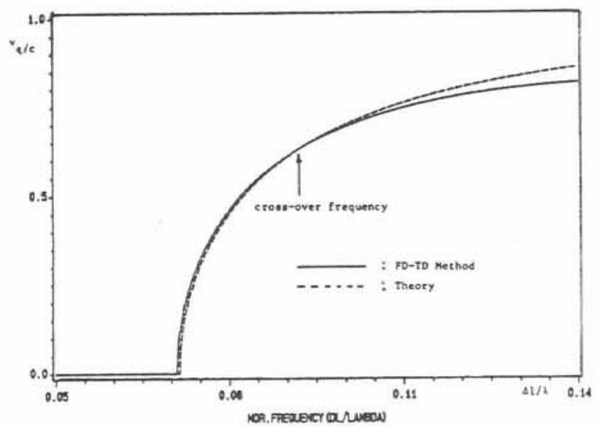

Fig. 4 a) Group Velocity Characteristics of a rectangular W/G.

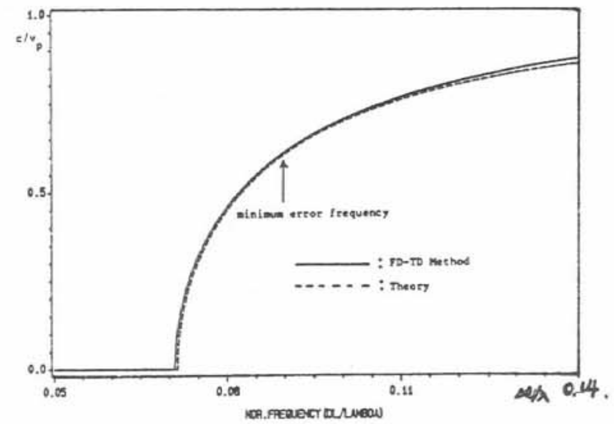

Fig. 4 b) Phase Velocity Characteristics of a rectangular W/G.