THE MATHEMATIC THEORY AND PHYSICAL CONCEPT OF A NEW METHOD FOR SOLVING MAXWELL'S EQUATION

Li Chunbao

Department of Electronic Engineering
Dalian Maritime University, China

I. The generalized Poisson equation of fields with vorticity

At 1812, Siméon Denis Poisson published the mathematic theory of static electricity and created the basics of the theory of static electricity. The Poisson equation is applicable for fields with divergence only. Now, the generalized Poisson equation is applicable for fields with both divergence and vorticity.

we inspect the general field equations with divergence and vorticity,

$$\nabla \times \vec{f} = \vec{w} \tag{1}$$

$$\nabla \cdot \vec{f} = p \tag{2}$$

In orthogonal curvilinear coordinates, equation (1) can be rewriten as

$$\begin{aligned}\vec{e}_i \frac{1}{h_j h_k}\left[\frac{\partial(h_k f_k)}{\partial u_j} - \frac{\partial(h_j f_j)}{\partial u_k}\right] + \vec{e}_j \frac{1}{h_k h_i}\left[\frac{\partial(h_i f_i)}{\partial u_k} - \frac{\partial(h_k f_k)}{\partial u_i}\right] \\ + \vec{e}_k \frac{1}{h_i h_j}\left[\frac{\partial(h_j f_j)}{\partial u_i} - \frac{\partial(h_i f_i)}{\partial u_j}\right] = \vec{e}_i w_i + \vec{e}_j w_j + \vec{e}_k w_k\end{aligned} \tag{3}$$

That is,

$$\frac{1}{h_j h_k}\left[\frac{\partial(h_k f_k)}{\partial u_j} - \frac{\partial(h_j f_j)}{\partial u_k}\right] = w_i \tag{4}$$

$$\frac{1}{h_k h_i}\left[\frac{\partial(h_i f_i)}{\partial u_k} - \frac{\partial(h_k f_k)}{\partial u_i}\right] = w_j \tag{5}$$

$$\frac{1}{h_i h_j}\left[\frac{\partial(h_j f_j)}{\partial u_i} - \frac{\partial(h_i f_i)}{\partial u_j}\right] = w_k \tag{6}$$

Equation (2) can be rewritten as

$$\frac{1}{h_i h_j h_k}\left[\frac{\partial(h_j h_k f_i)}{\partial u_i} + \frac{\partial(h_k h_i f_j)}{\partial u_j} + \frac{\partial(h_i h_j f_k)}{\partial u_k}\right] = p \tag{7}$$

Where u_i, u_j, u_k, are the variables of the orthogonal curvilinear coordinates, $\vec{e}_i, \vec{e}_j, \vec{e}_k$, are unit vectors and h_i, h_j, h_k, are scalar factors. From (5),

$$\frac{\partial(h_i f_i)}{\partial u_k} = h_i h_k w_j + \frac{\partial(h_k f_k)}{\partial u_i}$$

$$f_i = \frac{1}{h_i}\int h_i h_k w_j du_k + \frac{1}{h_i}\int \frac{\partial(h_k f_k)}{\partial u_i} du_k \tag{8}$$

we take the integration constant as zero for a particular solution.

Substitute (8) into (6), we have

$$f_j = \frac{1}{h_j}\int h_i h_j w_k du_i + \frac{1}{h_j}\int\int \frac{\partial h_i h_k}{\partial u_j} du_k du_i + \frac{1}{h_j}\int \frac{\partial (h_k f_k)}{\partial u_j} du_k \tag{9}$$

Substitute (8) and (9) into (7), we have

$$\frac{\partial}{\partial u_i}\cdot\frac{h_j h_k}{h_i}\int\frac{\partial (h_k f_k)}{\partial u_i} du_k + \frac{\partial}{\partial u_j}\cdot\frac{h_k h_i}{h_j}\int\frac{\partial (h_k f_k)}{\partial u_j} du_k + \frac{\partial}{\partial u_k}\cdot\frac{h_i h_j}{h_k}\int\frac{\partial (h_k f_k)}{\partial u_k} du_k$$

$$= \int\frac{\partial}{\partial u_j}\cdot\frac{h_k h_i}{h_j}(h_j h_k w_i) du_k - \int\frac{\partial}{\partial u_i}\cdot\frac{h_j h_k}{h_i}(h_i h_k w_j) du_k + h_i h_j h_k p$$

$$\therefore \quad \nabla^2 \int h_k f_k du_k = p + \frac{1}{h_i h_j h_k}\left[\int\frac{\partial}{\partial u_j}\cdot\frac{h_k h_i}{h_j}(h_j h_k w_i) du_k - \int\frac{\partial}{\partial u_i}\cdot\frac{h_j h_k}{h_i}(h_i h_k w_j) du_k\right]$$

Define

$$\int h_k f_k du_k = -\psi \tag{10}$$

ψ is called the scalar potential of fields with both divergence and vorticity, or it may be called the synthetic scalar potential of fields with divergence and vorticity. We shall prove that the usual scalar potential of the static electric field is the special case of the synthetic scalar potential when the vorticity of the fields is zero.

From (10), we have

$$f_k = \frac{-1}{h_k}\frac{\partial \psi}{\partial u_k} \tag{11}$$

and

$$\nabla^2\psi = -p - \frac{1}{h_i h_j h_k}\left[\int\frac{\partial}{\partial u_j}\cdot\frac{h_k h_i}{h_j}(h_j h_k w_i) du_k - \int\frac{\partial}{\partial u_i}\cdot\frac{h_j h_k}{h_i}(h_i h_k w_j) du_k\right] \tag{12}$$

$$= -p + Q$$

where

$$Q = \frac{-1}{h_i h_j h_k}\left[\int\frac{\partial}{\partial u_j}\cdot\frac{h_k h_i}{h_j}(h_j h_k w_i) du_k - \int\frac{\partial}{\partial u_i}\cdot\frac{h_j h_k}{h_i}(h_i h_k w_j) du_k\right] \tag{13}$$

Q is called the Pseudo-divergence of general fields and can be transformed from the vorticity of the fields according to the formula (13).

Equation (12) is the generalized Poisson equation of fields with vorticity. It is shown clearly that the generalized Poisson equation (12) recedes to the usual Poisson equation in static electric fields when the vorticity of the fields is zero, that is

$$\nabla^2\psi = -p$$

Then the usual Poisson equation is the special case of the generalized Poisson equation (12). The subscripts i, j, k may be cycled by the right hand rule.

II. The application of the generalized Poisson equation for solving the Maxwell equation

1. Using the generalized Poisson equation to get the solution of the general field equations with divergence and vorticity

Since

$$\vec{f} = \vec{e}_i f_i + \vec{e}_j f_j + \vec{e}_k f_k$$

From (8), (9), (11), we have

$$\vec{f} = -\nabla\psi + \vec{e}_i \frac{1}{h_i} \int h_j h_k w_j du_k - \vec{e}_j \frac{1}{h_j} \int h_j h_k w_i du_k \tag{14}$$

This is the particular solution of the general vector fields. It is easily seen that the field intensity $\vec{f}$ recedes to $\vec{f} = -\nabla\psi$, which is the usual form of the field with no curl. If the vorticity $\vec{w}$ and divergence ρ are distributed in a finite region and the volume V recedes to infinity, the particular solution (14) is the complete solution of the general vector field. If the region V is finite, the solution (14) is only part of the complete solution. In this case, it is well known that the another part is the solution of Neumann boundary value problem, and the solving process of it can be found in published books.

2. The new method for solving Maxwell equation

To inspect the Maxwell equation

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t} = \vec{\varepsilon} \tag{15}$$

$$\nabla \cdot \vec{D} = \rho \tag{16}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{\eta} \tag{17}$$

$$\nabla \cdot \vec{B} = 0 \tag{18}$$

It may be divided into two groups: one group consists of equations (15) and (16), the other consists of (17) and (18). From (15) and (16), using (12) and marking with a subscript "e", we have

$$\triangle\psi_e = \frac{-\rho}{\varepsilon_0} + \int \left(\frac{\partial e_z}{\partial y} - \frac{\partial e_y}{\partial z} \right) dx$$

$$= \frac{-\rho}{\varepsilon_0} - \mu_0 \int \left(\frac{\partial^2 H_z}{\partial t \partial y} - \frac{\partial^2 H_y}{\partial t \partial z} \right) dx$$

$$\triangle\psi_e = \frac{-\rho}{\varepsilon_0} - \mu_0\frac{\partial}{\partial t}\int\left(J_z + \varepsilon_0\frac{\partial E_z}{\partial t}\right)dx$$

$$= \frac{-\rho}{\varepsilon_0} - \mu_0\int\frac{\partial J_z}{\partial t}dx + \mu_0\varepsilon_0\frac{\partial^2\psi_e}{\partial t^2}$$

we get

$$\triangle\psi_e - \mu_0\varepsilon_0\frac{\partial^2\psi_e}{\partial t^2} = \frac{-\rho}{\varepsilon_0} - \mu_0\int\frac{\partial J_z}{\partial t}dx \tag{19}$$

Similarly for (17) and (18), we use the subscript "m", we get

$$\triangle\psi_m - \mu_0\varepsilon_0\frac{\partial^2\psi_m}{\partial t^2} = \int\left(\frac{\partial J_y}{\partial x} - \frac{\partial J_x}{\partial y}\right)dz \tag{20}$$

(19) and (20) are the D'alembert equations which may be used to obtain $\vec{E}$ and $\vec{H}$ of the Maxwell equation. From (14), we have the following formulas through simple calculations.

$$\begin{cases} E_x = -\dfrac{\partial\psi_e}{\partial x} \\ E_y = \mu_0\displaystyle\int\frac{\partial^2\psi_m}{\partial z\partial t}dx - \frac{\partial\psi_e}{\partial y} \\ E_z = -\mu_0\displaystyle\int\int\frac{\partial J_z}{\partial t}dzdx + \mu_0\varepsilon_0\int\frac{\partial^2\psi_e}{\partial t^2}dz - \mu_0\int\frac{\partial^2\psi_m}{\partial y\partial t}dx - \frac{\partial\psi_e}{\partial z} \end{cases} \tag{21}$$

$$\begin{cases} H_x = \displaystyle\int J_y dz + \mu_0\varepsilon_0\int\frac{\partial^2\psi_m}{\partial t^2}dx - \varepsilon_0\int\frac{\partial^2\psi_e}{\partial y\partial t}dz - \frac{\partial\psi_m}{\partial x} \\ H_y = -\displaystyle\int J_x dz + \varepsilon_0\int\frac{\partial^2\psi_e}{\partial x\partial t}dz - \frac{\partial\psi_m}{\partial y} \\ H_z = -\dfrac{\partial\psi_m}{\partial z} \end{cases} \tag{22}$$

We conclude that equations (19) and (20) are more symmetrical and the new method is more convenient through using the generalized Poisson equation.

References

(1) Li Chunbao, To Construct a Vector Field with Given Curl Function and Divergence Function, Applied Math. & Mechan. Vol. 2, No. 5, (1981), 607-612

(2) Li Chunbao & Wang, Proceed. of the 1985 ISAP, Japan, P.277

(3) Cochin, N. E. Vector Calculus & Tensor Calculus, Academy of Science USSR, 1965, P. 214 (in Russian)

(4) Coffin J. G., Vector Analysis, (1911) , 160

(5) Fluid Mechanics, Edited by Dept. of Mathematics, Fudan University, China (1960)

(6) Popovic B. D., IEE Proc. 128 ,Part A, No. 1,1981, 47