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A COMPARISON OF THE KENNAUGH MATRIX AND THE COVARIANCE MATRIX APPROACH IN RADAR POLRIMETRY

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Abstract. A comparison between the Kennaugh matrix and the covariance matrix formulation of second-order moments radar polarimetry reveals interesting relationships between power and variance aspects of radar scattering from random targets.

Introduction. In radar and in optical polarimetry there exist two different methods to characterize the polarimetric scattering properties of plane fully polarized electromagnetic waves by random targets by second-order moments, the Kennaugh matrix formulation and the covariance matrix analysis. They are generally considered to be independent, although formally they involve the same second-order moments. The Kennaugh approach is used for finding solutions for optimal power transfers whereas the covariance matrix is used for entropy and variance considerations. In the following we restrict ourselves to backscatter radar polarimetry for which we refer to Boerner et al [1] for a detailed account. The four elements of the Sinclair back scatter matrix S in the common transmit/receive $\{x, y\}$ - or $\{H, V\}$ - polarization basis are correlated random variables where t stands for ensemble values

$$S(t) = \begin{bmatrix} S_{xx}(t) & S_{xy}(t) \\ S_{yx}(t) & S_{yy}(t) \end{bmatrix}$$

It will be assumed that the elements of the Sinclair matrix $S(t)$ at a fixed but arbitrary instant of time or space are random variables of a stationary and/or homogeneous stochastic processes. Assuming ergodicity ensemble averages can be replaced by time averages and are denoted by sharp brackets $< ... >$. For sake of simplicity we take $< S(t) > = 0$ which implies removing of means.

Second-order moments of these random variables are taken into account by forming either one of these expressions

$$S(t) \quad \rightarrow \quad \begin{cases} K = < S(t) \otimes S^*(t) > & \text{pre-Kennaugh matrix} \\ C = < \mathrm{vec}\, S(t) \mathrm{vec}^\dagger S(t) > & \text{covariance matrix} \end{cases}$$

where the symbol $\otimes$ denotes the Kronecker product and the 'vec' operator indicates the subsequent stacking of column vectors, see Horn/Johnson [2]. The dagger symbol † denotes Hermitian conjugation.

The Kennaugh matrix. The pre-Kennaugh matrix K arises in the following way. The random electromagnetic field $\vec{E}^s(t)$ resulting from scattering of the incident fully polarized wave $\vec{E}^i$ by a randomly distributed target is combined with its complex conjugate as follows

$$\left.\begin{aligned} \vec{E}^s(t) &= S(t)E^i \\ \vec{E}^{s*}(t) &= S^*(t)E^{i*} \end{aligned}\right\} \quad \rightarrow \quad \vec{E}^s(t) \otimes \vec{E}^{s*}(t) = (S(t)\vec{E}^i) \otimes (S^*(t)E^{i*}) = (S(t) \otimes S^*(t))(\vec{E}^i \otimes \vec{E}^{i*}).$$

Averaging leads to

$$< \vec{E}^{s}(t) \otimes \vec{E}^{s*}(t) > = < S(t) \otimes S^{*}(t)) > (E^{i} \otimes E^{i*}) = K(E^{i} \otimes E^{i*}).$$

The pre-Kennaugh matrix K reads explicitly

$$K = < S(t) \times S^{*}(t) > = < \begin{bmatrix} |S_{xx}(t)|^2 & S_{xx}(t)S^{*}_{xy}(t) & S_{xy}(t)S^{*}_{xx}(t) & |S_{xy}(t)|^2 \\ S_{xx}(t)S^{*}_{yx}(t) & S_{xx}(t)S^{*}_{yy}(t) & S_{xy}(t)S^{*}_{yx}(t) & S_{xy}(t)S^{*}_{yy}(t) \\ S_{yx}(t)S^{*}_{xx}(t) & S_{yx}(t)S^{*}_{xy}(t) & S_{yy}(t)S^{*}_{xx}(t) & S_{yy}(t)S^{*}_{xy}(t) \\ |S_{yx}(t)|^2 & S_{yx}(t)S^{*}_{yy}(t) & S_{yy}(t)S^{*}_{yx}(t) & |S_{yy}(t)|^2 \end{bmatrix} > = \begin{bmatrix} K_1 & K_3 \\ K_2 & K_4 \end{bmatrix}$$

where K_i $(i = 1,...,4)$ are 2×2 sub-matrices. Multiplication with the 4×4 matrix Q

$$Q < E \otimes \vec{E}^{*} > = Q \begin{bmatrix} <|E_x|^2> \\ <E_x E_y^{*}> \\ <E_y E_x^{*}> \\ <|E_y|^2> \end{bmatrix} = \begin{bmatrix} <|E_x|^2> + <|E_y|^2> \\ <|E_x|^2> - <|E_y|^2> \\ 2 < \mathrm{Re} E_x^{*} E_y > \\ 2 < \mathrm{Im} E_x^{*} E_y > \end{bmatrix} = g = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & j & -j & 0 \end{bmatrix}$$

leads to the (real) Stokes vectors g of the scattered and incident electric fields.. The Kennaugh matrix K_e, see Mott [3], is defined by

$$K_e = \frac{1}{2} Q^{*} < S(t) \otimes S^{*}(t) > Q^{\dagger} = \frac{1}{2} Q^{*} K Q^{\dagger} = \frac{1}{2} DQKQ^{\dagger} = DQKQ^{-1}$$

with the diagonal matrix $D = \mathrm{diag}[1,1,1,-1]$. The matrix D arises from the application of the voltage equation to power optimization problems. Actually it should be considered as the time reversal operation in polarimetrric Minkowshi space taking into account that the incident and the scattered wave in backscatter radar polarimetry propagate in opposite directions, see Lüneburg [5]. It is worthwhile to point out that the Kennaugh matrix is real and symmetric if K (or even S for a point target) is symmetric. These expressions can be found also from Wolf's coherency matrix [6].

The covariance matrix. We introduce the so-called polarimetric target feature vector

$$\vec{k}(t) = \mathrm{vec}\, S(t) = \begin{bmatrix} k_1(t) \\ k_2(t) \\ k_3(t) \\ k_4(t) \end{bmatrix} = \begin{bmatrix} S_{HH}(t) \\ S_{VH}(t) \\ S_{HV}(t) \\ S_{VV}(t) \end{bmatrix}.$$

which is just a straightforward way of writing the random elements in a convenient form, Then the explicit form of covariance matrix C is given by $C = < \vec{k}(t) \vec{k}^{\dagger}(t) >$ or

$$C=\begin{bmatrix} <|S_{HH}(t)|^2> & <S_{HH}(t)S_{VH}^*(t)> & <S_{HH}(t)S_{HV}^*(t)> & <S_{HH}(t)S_{VV}^*(t)> \\ <S_{VH}(t)S_{HH}^*(t)> & <|S_{VH}(t)|^2> & <S_{VH}(t)S_{HV}^*(t)> & <S_{VH}(t)S_{VV}^*(t)> \\ <S_{HV}(t)S_{HH}^*(t)> & <S_{HV}(t)S_{VH}^*(t)> & <|S_{HV}(t)|^2> & <S_{HV}(t)S_{VV}^*(t)> \\ <S_{VV}(t)S_{HH}^*(t)> & <S_{VV}(t)S_{VH}^*(t)> & <S_{VV}(t)S_{HV}^*(t)> & <|S_{VV}(t)|^2> \end{bmatrix}=\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}.$$

By definition the 4×4 covariance matrix C is Hermitian positive semi-definite. We note the following relation between the pre-Kennaugh matrix K and the covariance matrix $C: C_i=\mathrm{vec}^T K_i \quad (i=1,...,4)$. Another column vector ordering uses the Pauli spin matrices. Both forms are related by unitary similarity and hence have the same norm and eigenvalues. Covariance or correlation matrices describe second-order moments of multivariate (often Gaussian) joint probability density functions. In radar polarimetry the application of the covariance matrix is known as target decomposition theory, see Cloude [7]. This is however a special application of the widely-known Principle Component Analysis (PCA) from general multivariate statistics, see Jolliffe [4]. The main purpose of PCA is to convert the correlated random variable $k_i(t)$ into new uncorrelated (not necessary independent) variables and order the new variables according to their variances.

Being Hermitian positive semi-definite the covariance matrix C can be unitarily diagonalized

$$U^{-1}CU=\Lambda\equiv\mathrm{diag}[l_1,\,l_2,\,l_3,\,l_4] \quad \text{with} \quad U=[\hat{x}_1,\hat{x}_2,\hat{x}_3,\hat{x}_4].$$

We introduce new random variables by means of the linear relation

$$\vec{Z}(t)=\begin{bmatrix} z_1(t) & z_2(t) & z_3(t) & z_4(t)\end{bmatrix}^T=U^{\dagger}\vec{k}(t)=\begin{bmatrix}\hat{x}_1^{\dagger}\vec{k}(t) & \hat{x}_2^{\dagger}\vec{k}(t) & \hat{x}_3^{\dagger}\vec{k}(t) & \hat{x}_4^{\dagger}\vec{k}(t)\end{bmatrix}^T \quad \text{or} \quad \vec{k}(t)=U\vec{Z}(t).$$

with real nonnegative eigenvalues $0\le l_4\le l_3\le l_2\le l_1$ and orthonormal eigenvectors $\hat{x}_i$

$$C\hat{x}_i=l_i\hat{x}_i, \quad \hat{x}_i^{\dagger}\hat{x}_j=d_{ij} \quad (i,j=1,2,3,4).$$

The new random vector components $z_i(t) \; (i=1,2,3,4)$ of the target feature vector $\vec{Z}(t)$ are called the principal components (PC's), see Jolliffe [4]. They are uncorrelated (but not necessarily independent) and their variance is equal to the corresponding eigenvalue of C

$$<z_i(t)z_j^*(t)>=\hat{x}_i^{\dagger}<\vec{k}\vec{k}^{\dagger}>\hat{x}_j=\hat{x}_i^{\dagger}C\hat{x}_j=l_j d_{ij} \quad (i,j=1,2,3,4).$$

The eigenvectors $\hat{x}_i \; (i=1,2,3,4)$ of the covariance matrix C are called the vectors of coefficients or loadings for the i-th principal component $z_i(t)$. The spectral decomposition of C is given by $C=U\Lambda U^{\dagger}=\sum_{i=1}^{4} l_i\vec{x}_i\vec{x}_i^{\dagger}$. The four 4×4 matrices $\hat{x}_i\hat{x}_i^{\dagger}$ all have rank 1 and $\mathrm{trace}(\hat{x}_i\hat{x}_i^{\dagger})=1$. Reversing the 'vec' operation the $\hat{x}_i$'s can be interpreted as 2×2 point targets S_i with $\mathrm{span}\,S_i=1$

$$\hat{x}_i=\mathrm{vec}\,S_i=\begin{bmatrix}x_{i1} & x_{i2} & x_{i3} & x_{i4}\end{bmatrix}^T \quad \leftrightarrow \quad S_i=\begin{bmatrix} x_{i1} & x_{i3} \\ x_{i2} & x_{i4}\end{bmatrix} \quad (i=1,2,3,4)$$

and hence by making use of the relation $\vec{k}(t)=U\vec{Z}(t)$

$$S(t)=\begin{bmatrix} S_{HH}(t) & S_{HV}(t) \\ S_{VH}(t) & S_{VV}(t) \end{bmatrix}=\sum_{i=1}^{4}\begin{bmatrix} x_{i1} & x_{i3} \\ x_{i2} & x_{i4} \end{bmatrix} z_i(t)=\sum_{i=1}^{4} S_i z_i(t).$$

This is the expansion of the samples of the Sinclair matrix $S(t)$ into four point targets with random coefficients. It should be stressed that the new random variables $z_i(t)$ are uncorrelated and that the basic targets S_i are orthonormal in the sense

$$(\mathrm{vec}S_i)^{\dagger}\mathrm{vec}S_j=\hat{x}_i^{\dagger}\hat{x}_j=\sum_{k=1}^{4} x_{ik}^{*}x_{jk}=\sum_{k=1}^{4}|x_{ij}|^2 d_{ij}=\|\hat{x}_i\|^2\, d_{ij}=d_{ij}.$$

The average Graves' power matrix is Hermitian positive semi-definite and can be expanded according to $\bar{G}=\,<S^{\dagger}(t)S(t)>\,=\sum_{i=1}^{4} l_i S_i^{\dagger}S_i=\sum_{i=1}^{4} l_i G_i$ with $\mathrm{trace}\,\bar{G}=\sum_{i=1}^{4} l_i$. $\bar{G}$ is a weighted sum of the basic Hermitian positive semi-definite Graves' power matrices $G_i=S_i^{\dagger}S_i$.

Using the fact that the variables $z_i(t)$ are uncorrelated we obtain the expansion

$$S(t)\otimes S^{*}(t)=\sum_{i,\,j=1}^{4} z_i(t)\,z_j^{*}(t)\,(S_i\otimes S_j^{*})\quad\rightarrow\quad <S(t)\otimes S^{*}(t)>\,=\sum_{i=1}^{4} l_i(S_i\otimes S_i^{*})=\sum_{i=1}^{4} l_i K_i,$$

and therefore

$$K_e=\sum_{i=1}^{4} l_i K_{e,i}\quad\text{with}\quad K_{e,i}=\frac{1}{2}Q^{*}(S_i\otimes S_i^{*})Q^{\dagger}=\frac{1}{2}Q^{*}K_iQ^{\dagger},$$

i.e., the Kennaugh matrix is the sum of 4 elementary Kennaugh matrices of point targets with weights l_i . With this analysis, we have shown the intimate connection between the matrices $S(t)$, K_e , K and C .

References

[1] Boerner, W.-M., H. Mott, E. Lüneburg, B. Brisco, R. Brown, J. S. Paterson (authors); S. R. Cloude, E. Krogager, J. S. Lee, D. L. Schuler, J. J. van Zyl, D. Randall, P. Budkewitsch (contributing authors), *Polarimetry in Remote Sensing: Basic and Applied Concepts*, Chapter 5 in R. A. Reyerson, ed. "The Manual of Remote Sensing", 3rd Edition, American Society of Photogrammetry and Remote Sensing, ASPRS Publishing, Bethesda, MD, 1998.

[2] Horn, R. A. and Ch. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991.

[3] Mott, H., *Antennas for Radar and Communications: A Polarimetric Approach*, John Wiley \& Sons, New York 1992.

[4] Jolliffe, I.T., Principal Component Analysis, New York, Springer, 2002.

[5] Lüneburg, E., Principles of Radar Polarimetry, *IEICE Transactions on Electronics (Special Issue on Electromagnetic Theory), vol. E78-C, 10* (1995) 1339-1345.

[6] Wolf, E., 1959, "Coherence Properties of Partially Polarized Electromagnetic Radiation", *Nuovo Cimento*, vol. XIII (6), pp. 1165-1181.

[7] Cloude, S.R., Uniqueness of target decomposition theorems in radar polarimetry, Proceedings of the NATO Advanced Research Workshop on *Direct and Inverse Methods in Radar Polarimetry*, W.-M. Boerner et al (eds), Bad Windsheim, Germany, September 18-24, 1988; Kluwer Academic Publishers, Dordrecht 1992; NATO ASI Series C-vol. 350, 1938p. pp. 267-296.