AN ANALYTIC SOLUTION OF THE TWO-FREQUENCY MOMENT EQUATION IN AN INHOMOGENEOUS TURBULENT MEDIUM

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1. Introduction

In this paper, an analytic solution of the two frequency moment equation is obtained for spherical wave propagation in an inhomogeneous turbulent medium using the quadratic structure function approximation.

In homogeneous turbulence, analytic solutions were given for plane and spherical waves[1][2]. Another analytic solution was obtained for spherical wave propagation in free space - homogeneous turbulence - free space[3]. They are solutions of the two-frequency moment equation under the Markov approximation and the quadratic structure function approximation. On the other hand, the two-frequency mutual coherence function was analyzed for spherical wave propagation using extended Huygens-Fresnel principle[4].

2. Inhomogeneous Turbulent Medium

An inhomogeneous random medium is assumed to be described by the dielectric constant ε, the magnetic permeability μ, and the electric conductivity σ, which are expressed as

$$\varepsilon=\varepsilon_0[1+\delta\varepsilon(\mathbf{r},z)] \ , \ \mu=\mu_0 \ , \ \sigma=0 \tag{1}$$

Here, $\mathbf{r}=\mathbf{i}x+\mathbf{j}y$ (i,j denote the unite vectors of x and y coordinates), ε_0 and μ_0 are constant, and $\delta\varepsilon(\mathbf{r},z)$ is a random function with properties

$$<\delta\varepsilon(\mathbf{r},z)>=0 \ , \ <\delta\varepsilon(\mathbf{r}_1,z_1)\delta\varepsilon(\mathbf{r}_2,z_2)>=B(\rho,z_+,z_-) \tag{2}$$

where the angular brackets denotes the ensemble average, $\rho=\mathbf{r}_1-\mathbf{r}_2$, $z_+=(z_1+z_2)/2$, and $z_-=z_1-z_2$. Moreover, it is assumed that for any z,

$$kl(z)\gg 1 \ , \ B(z)\equiv B(0,z,0)\ll 1 \tag{3}$$

where k is the wavenumber in free space and $B(z),l(z)$ are the local intensity and the scale size of the turbulence, respectively; and their changes are slow with z. Under the condition (3), the scalar approximation is valid; In addition, the forward scattering and small angle approximation can be applied, and the assumption of only the z-direction inhomogeneity of the medium fluctuation becomes valid.

3. Propagation and Detection of Pulse Waveform

In general, a pulse wave of the carrier angular frequency ω_0 may be expressed at a point $(\mathbf{r},z)$ as follows:

$$v(\mathbf{r},z,t)=Re[u(\mathbf{r},z,t)exp(-j\omega_0 t)] \tag{4}$$

Here, $Re[\cdot]$ denotes the real part of $[\cdot]$, and $u(\mathbf{r},z,t)$ is the complex envelope.

Suppose that a pulse wave of complex envelope $u_{in}(\mathbf{r},z,t)$, radiated at the transmitting point $(0,0)$, propagates through the inhomogeneous random medium. Then the Fourier transform of the complex envelope at the receiving point $(\mathbf{r},z)$, written as $u(\mathbf{r},z,\omega)$, is given as a solution of the following equation.

(5)
$$u(\mathbf{r},z,\omega)=u_{in}(\mathbf{r},z,\omega)+\int_{-\infty}^{\infty} d\mathbf{r}' \int_0^z dz' G(\mathbf{r}-\mathbf{r}',z-z',\omega)k^2\delta\varepsilon(\mathbf{r}',z')u(\mathbf{r}',z',\omega)$$

where $G(\mathbf{r},z,\omega)$ is the Green function expressed as

(6)
$$G(\mathbf{r},z,\omega)=\frac{1}{4\pi z}exp[jk(z+\frac{r^2}{2z})] \quad , \quad z>0$$

and $k=(\omega_0+\omega)/c$; $c=1/\sqrt{\varepsilon_0\mu_0}$.

The pulse wave $v(\mathbf{r},z,t)$ received by an antenna passes though the front-end electronics with the transfer function $H(\omega_0+\omega)$ and enters a square law detector. The instantaneous power in the pulse after detection is proportional to $v^2(\mathbf{r},z,t)$, and contains slowly and rapidly oscillating terms. By removing the rapidly oscillating term, the output response is proportional to $I=|u(\mathbf{r},z,t)|^2$. Therefore, through a time averaging process, we can obtain the average intensity of the pulse envelope. The average intensity is expressed in terms of the ensemble average $\langle I\rangle$ on the ergodic hypothesis. That is,

(7)
$$<I>=\frac{1}{(2\pi)^2}\int d\omega_1 \int d\omega_2 \left\{H(\omega_0+\omega_1)H^*(\omega_0+\omega_2)M_{11}(\mathbf{r}_1,\mathbf{r}_1,z,\omega_0,\omega_1,\omega_2)exp[j(\omega_1-\omega_2)t]\right\}$$

where the directional property of the receiving antenna is neglected, and

(8)
$$M_{11}(\mathbf{r}_1,\mathbf{r}_2,z,\omega_0,\omega_1,\omega_2)\equiv <u(\mathbf{r}_1,z,\omega_1)u^*(\mathbf{r}_2,z,\omega_2)>$$

4.Two-Frequency Moment

The moment(8) satisfies the two-frequency moment equation[5] without the Markov approximation:

(9)
$$\left\{\frac{\partial}{\partial z}-j\frac{1}{2}\left(\frac{\nabla_1^2}{k_1}-\frac{\nabla_2^2}{k_2}\right)-j(k_1-k_2)+\frac{1}{4}\int_0^z dz'[(k_1^2+k_2^2)B(0,z-\frac{z'}{2},z')-2k_1k_2B(\mathbf{r},z-\frac{z'}{2},z')]\right\}M_{11}=0$$

(10)
$$M_{11}|_{z=0}=u_{in}(\mathbf{r}_1,0,\omega_1)u^*_{in}(\mathbf{r}_2,0,\omega_2)$$

where $k_i=(\omega+\omega_i)/c$, $\nabla_i=\mathbf{i}\partial/\partial x_i+\mathbf{j}\partial/\partial y_i$, for $i=1,2$.

Transforming $\mathbf{r}_1$, $\mathbf{r}_2$, k_1, k_2 into $\mathbf{r}_+$, $\mathbf{r}_-$, k_s, k_d:

(11)
$$\mathbf{r}_+=(\mathbf{r}_1+\mathbf{r}_2)/2 \quad , \quad \mathbf{r}_-=\mathbf{r}_1-\mathbf{r}_2 \quad , \quad k_s=(k_1+k_2)/2 \quad , \quad k_d=k_1-k_2$$

assuming $k_s\gg|k_d|$ and $k_s\simeq k_0=\omega_0/c$, and expressing the moment by

(12)
$$M_{11}=M(\mathbf{r}_-,z)\cdot u_{in}(\mathbf{r}_1,z,\omega_1)u^*_{in}(\mathbf{r}_2,z,\omega_2)exp\left[-\frac{k_d^2}{4}\int_0^z dz'B(0,z-\frac{z'}{2},z')\right],$$

we get the following equation for $M(\mathbf{r}_-,z)$ in the case of spherical wave propagation.

(13)
$$\left[\frac{\partial}{\partial z}+j\frac{k_d}{2k_0^2}\nabla_-^2+\frac{1}{z}(\mathbf{r}_-\cdot\nabla_-)+D_t(\mathbf{r}_-,z)\right]M(\mathbf{r}_-,z)=0$$

(14)
$$M(\mathbf{r}_-,0)=1$$

where

(15) $$D_t(\mathbf{r}_-,z)=\frac{k_0^2}{4}\int_0^z D(\mathbf{r}_-,z-\frac{z'}{2},z')dz'$$

in which $D(\mathbf{r}_-,z_+,z_-)=2[B(0,z_+,z_-)-B(\mathbf{r}_-,z_+,z_-)]$.

In such a strongly turbulent medium that the quadratic structure function approximation may be valid, (15) is approximately expressed in terms of

(16) $$D_t(\mathbf{r}_-,z)=D_0 k_0^2\frac{B(z)}{l(z)}\mathbf{r}_-^2$$

where D_0 is a dimensionless constant; for example, $D_0=\sqrt{\pi}/4$ for $B(\mathbf{r},z_+,z_-)=B(z_+)exp[-(\mathbf{r}^2+z_-^2)/l^2(z_+)]$. Equation(16) shows that the effect of inhomogeneity of turbulence is emphasized generally because $l(z)$ becomes short with increasing of $B(z)$.

When $B(z)/l(z)$ in (16)is expressed as

(17) $$B(z)/l(z)=(B_0/l_0)\cdot(z/L)^n$$

where L is a normalization distance and n is a number, then the solution of (13) can be given by

(18) $$M(\mathbf{r}_-,z)=exp\left[h(z,\omega_d)\mathbf{r}_-^2-j\frac{2k_d}{k_0^2}\int_0^z h(z',\omega_d)dz'\right]$$

where

(19) $$h(z,w_d)=\frac{1}{A_1}\frac{1}{P(z)}\frac{dP(z)}{dz}\ ;\ P(z)=\frac{1}{\sqrt{z}}[c_1J_{-\nu}(z_\nu)+c_2J_\nu(z_\nu)]\ ,\ \nu=\frac{1}{n+2}$$

(20) $$z_\nu=-2\nu z^{1/(2\nu)}\left[\frac{A_1A_2}{L^n}\right]^{1/2}\ ,\ A_1=j\frac{2k_d}{k_0^2}\ ,\ A_2=D_0k_0^2\frac{B_0}{l_0}$$

and c_1,c_2 are constant, determined by the boundary condition. In plane wave propagation, $(\mathbf{r}_-\cdot\nabla_-)/z$ in (13) is neglected. As a result, z_ν in (19) becomes $-z_\nu$ and $P(z)$ in (20) is rewritten as

(21) $$P(z)=\sqrt{z}\,[c_1J_{-\nu}(z_\nu)+c_2J_\nu(z_\nu)]$$

Other parameters A_1,A_2,ν is the same as in spherical wave propagation.

5.Mean Pulse Intensity

Using (7),(12),(18) and assuming $H(\omega_0+\omega_i)=1$, we can express the mean pulse intensity in the following form.

(22) $$<I>/|u_{in}(\mathbf{r},z,t)|^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega_d\,P(\omega_d)exp\left[-\frac{D_0}{2}k_d^2\frac{B(z)}{l(z)}-j\frac{2k_d}{k_0^2}\int_0^z h(z',\omega_d)dz'-j\omega_d t'\right]$$

where $\omega_d=\omega_1-\omega_2$, $\omega_s=(\omega_1+\omega_2)/2$, $t'=t-z/c$

(23) $$P(\omega_d)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega_s[u_{in}(\mathbf{r},z,\omega_1)u_{in}^*(\mathbf{r},z,\omega_2)]/|u_{in}(\mathbf{r},z,t)|^2$$

(24) $$\frac{k_d^2}{4}\int_0^z B(0,z-\frac{z'}{2},z')dz'\simeq\frac{D_0}{2}k_d^2\frac{B(z)}{l(z)}$$

were used. For a pulse waveform of the amplitude 1 and the pulse width $W=2T$, we have $P(\omega_d)=\pi sin(\omega_d T)/\omega_d$.

By changing the form of $B(z)/l(z)$ in (18), we can readily calculate the average pulse waveform from (24) in the propagation through the inhomogeneous turbulent medium. Figure 1 shows the average pulse intensity of plane and spherical waves in the up-link propagation, where $f_0=\omega_0/2\pi=6GHz$, $Zs=36{,}000km$, $L=300km$ are assumed. The $B_0/l_0=2\times10^{-14}$ means the *10%* fluctuation of the electron density for $l_0=100m$.

6.Concluding Remarks

The result obtained here is an extension of the previous results[1]~[3] to the inhomogeneous random medium. The effect of the inhomogeneity of fluctuation on the pulse waveform distortion becomes remarkable with increasing of the intensity and decreasing of the scale size. The equation for the mean pulse intensity may be applied to some practical cases.

Acknowledgment

This work was supported by a scientific research grant-in-aid(grant 61550244,1987) from the Ministry of Education, Science, and Culture of Japan, and by the KDD Engineering and Consulting Foundation.

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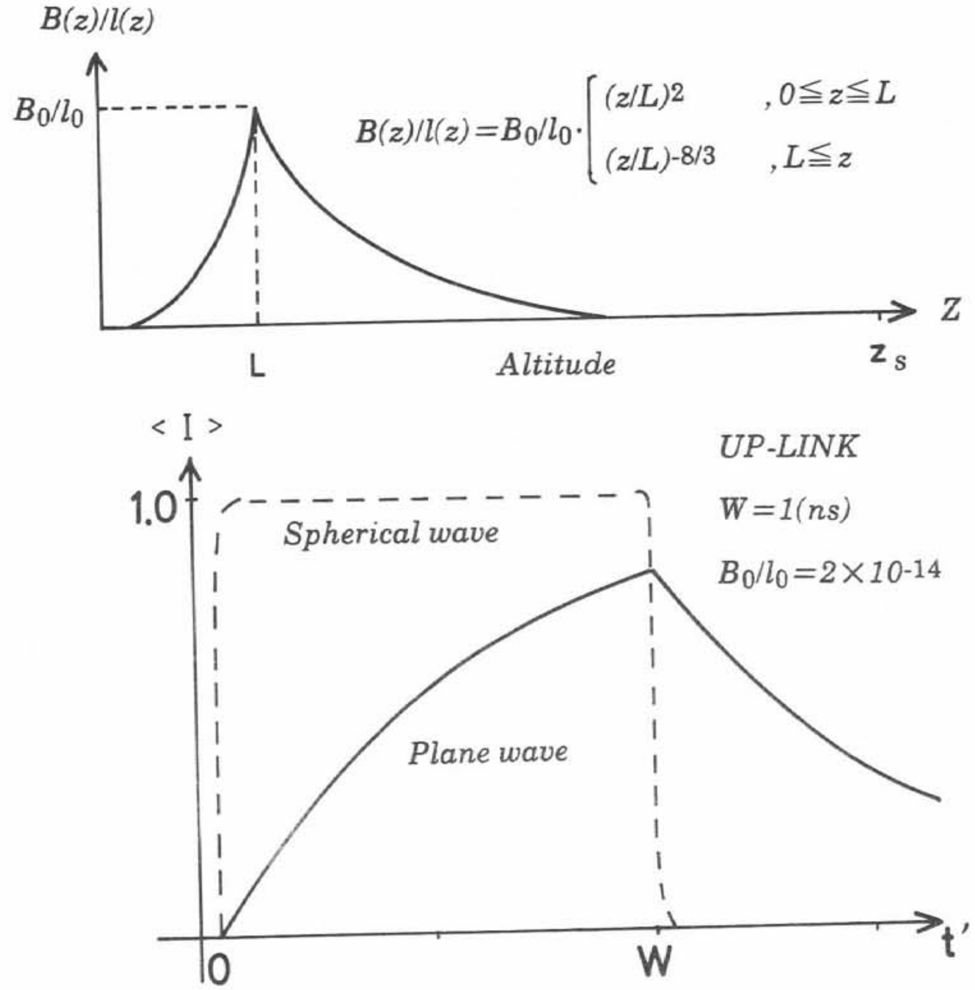

Fig. 1. The mean rectangular pulse waveform propergated through the inhomogeneous random medium.