SCATTERING OF WAVE FROM A RANDOM SPHERICAL SURFACE

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Abstract

The stochastic theory of a scattering from a random sphere is developed based on the theory of the rotation group. Several formulas are derived for the coherent scattering amplitude, the total and differential cross-sections for the coherent and incoherent scatterings and the stochastic version of the optical theorem. Numerical calculations are given for the Mie scattering from a slightly random spherical surface.

INTRODUCTION

In a previous paper[1] the two-dimensional (2D) scattering of a plane wave from a random cylindrical surface was treated using a stochastic functional method combined with a group-theoretic consideration related to the circular rotations. The present paper gives its extension to the 3D scattering from a random spherical surface which is assumed to be statistically homogeneous with respect to rotations. The treatment is closely related to the representation theory of the rotation group[2] in connection with the rotational homogeneity. As in the nonrandom case, the random wave field is expanded in terms of 'stochastic' spherical harmonics, each of which belongs to an irreducible vector space with respect to the rotational shift transformations of the random field. Mathematical manipulations can be greatly simplified by the group-theoretic treatment.

As shown in the cylindrical case [1], the effect of multiple scattering is small in the Mie scattering with small roughness, so that in this paper we deal with the single scattering approximation for a random sphere. Followings are brief accounts of the theoretical and numerical results for the random Mie scattering from a slightly random sphere.

RANDOM SPHERICAL SURFACE AND RANDOM WAVE FIELD

Let a random spherical surface with mean radius a be described by

$$r = a + f(\mathbf{r}), \quad < f(\mathbf{r}) >= 0 \tag{1}$$

where $\mathbf{r}$ denotes a point on the sphere, and $f(\mathbf{r})$ is assumed to be a homogeneous Gaussian random field on the sphere with the correlation function expressible in the form;

$$R(\mathbf{r}_1, \mathbf{r}_2) =< f(\mathbf{r}_1) f(\mathbf{r}_2) >= \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell+1) \left| F_\ell \right|^2 P_\ell(\cos\theta_{12}) \tag{2}$$

where $<\ >$ denotes the average, $|F_\ell|^2$ the 'power spectrum', and θ_{12} the angle between two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. The variance $\sigma =< f^2 >$ gives the parameter describing the roughness. The random wave field ψ satisfies the Helmhotz equation with the wave number k, as well as the Dirichlet or Neumann boundary condition on the random surface which can be approximated by the following equivalent boundary conditions at $r = a$:

$$[\psi + f \partial\psi/\partial r]_{r=a} = 0, \qquad \text{(Dirichlet)} \tag{3}$$

$$\left[\partial\psi/\partial r - \nabla f \cdot \nabla\psi + \partial^2\psi/\partial r^2\right]_{r=a} = 0 \qquad \text{(Neumann)} \tag{4}$$

As well known the plane wave can be decomposed into the sum of spherical waves each belonging to an irreducible space of weight ℓ of the rotation group G. We first obtain the scattered random wave field for the incidence of ℓ-th spherical wave, which can be given in terms of 'stochastic' spherical harmonics. Then we obtain the wave field for the plane wave incidence by means of the superposition over ℓ.

The total wave field for a plane wave incident in the direction of the polar axis (unit vector $\mathbf{e}_0$) can be written as the sum of the coherent part and the incoherent part;

$$\psi(\mathbf{r}) = \psi^{\rm c}(\mathbf{r}) + \psi^{\rm ic}(\mathbf{r}), \quad < \psi^{ic} >= 0 \tag{5}$$

Let $\mathbf{r} = (r, \theta, \phi)$ in the spherical coordinates. The coherent (average) part can be written

$$\psi^{\rm c}(\mathbf{r}) = \sum_{\ell=0}^{\infty} \imath^{\ell}(2\ell+1)\left[j_\ell(kr) + \alpha_\ell h_\ell^{(1)}(kr)\right] P_\ell(\cos\theta) \tag{6}$$

$$\alpha_\ell = \alpha_\ell^0 + A_\ell^{\rm c} \tag{7}$$

where α_ℓ^0 denotes the scattering coefficient for a smooth sphere ($\sigma = 0$),

$$\alpha_\ell^0 \equiv -j_\ell(ka)/h_\ell^{(1)}(ka) \qquad \text{(Dirichlet)} \tag{8}$$

$$\alpha_\ell^0 \equiv -j'_\ell(ka)/h_\ell^{(1)'}(ka) \qquad \text{(Neumann)} \tag{9}$$

and $A_\ell^{\rm c}$ gives the correction due to random surface. The incoherent (random) part can be written

$$\psi^{\rm ic}(\mathbf{r}) = \sum_{\ell=0}^{\infty} \imath^{\ell}(2\ell+1) \sum_{\ell'=0}^{\infty} \sum_{m=-L}^{L} H_m^{\ell\ell'}(kr) \mathrm{T}_{0m}^{\ell}(g) B_{\ell'}^{m}(\mathbf{r}) \tag{10}$$

$$H_m^{\ell\ell'}(kr) \equiv \sum_{n=-L}^{L} h_{mn}^{(1)\ell'\ell}(kr) A_n^{\ell\ell'}, \quad L \equiv \min(\ell, \ell') \tag{11}$$

where $\mathrm{T}_{0m}^{\ell}(g)$ denotes a generalized spherical function[2] ($\mathbf{r} = rg\mathbf{e}_0, g$:rotation), $h_{mn}^{(1)}(kr)$ the generalized spherical Hankel function defined by

$$h_{mn}^{(1)\ell'\ell}(kr) \equiv \sum_{\mathrm{L}=|\ell-\ell'|}^{\ell+\ell'} \imath^{L-\ell+\ell'}(-1)^{m+n} \left(\ell - m\ell' m|\ell\ell' L0\right) \left(\ell - n\ell' n|\ell\ell' L0\right) h_L^{(1)}(kr) \tag{12}$$

$(\ell - m\ell' m|\ell\ell' L0)$ being the Clebsch-Gordan coefficient. And $B_{\ell'}^{m}(\mathbf{r})$ in (10) gives the independent Gaussian variable with unit variance associated with the Gaussian random sphere.

$A_n^{\ell\ell'}$'s in (11), as well as $A_\ell^{\rm c}$ in (7), denote the expansion coefficients for the random wave field to be determined by solving the boundary condition (3) or (4).

STATISTICAL CHARACTERISTICS FOR SCATTERED WAVE FIELD

From the stochastic representation of the random wave field we can evaluate various statistical quantities for the random scattering.

Coherent scattering amplitude The coherent scattering part $\psi_s^{\rm c}(\mathbf{r})$ arising from the second terms in (6) has the asymptotic form as $r \to \infty$.

$$\psi_s^{\rm c}(\mathbf{r}) = \sum_{\ell=0}^{\infty} \imath^{\ell}(2\ell+1)\alpha_\ell h_\ell^{(1)}(kr) P_\ell(\cos\theta) \sim \frac{\mathrm{e}^{\imath kr}}{r}\Phi, \quad r \to \infty \tag{13}$$

$$\Phi(\theta) \equiv \frac{1}{k\imath} \sum_{\ell=0}^{\infty} (2\ell+1)\alpha_\ell P_\ell(\cos\theta) \tag{14}$$

We call $\Phi(\theta)$ the coherent scattering amplitude.

Total coherent power flow The total coherent power flow is given by

$$S_c = \lim_{r\to\infty} \frac{r^2}{k} \int_{S_3} \mathrm{Im}\left[\overline{\psi^c(\mathbf{r})}\frac{\partial \psi^c}{\partial r}\right] \mathrm{d}S = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\left[\mathrm{Re}\alpha_\ell + |\alpha_\ell|^2\right] \tag{15}$$

The term arising from the second term in the bracket gives the coherent power flow σ_c for the scattered wave.

Total incoherent power flow The total power flow due to ψ^{ic} is constant with probability 1 so that it is equal to its average:

$$\sigma_{ic} = \lim_{r\to\infty} \frac{r^2}{k} \int_S \mathrm{Im}\left\langle \overline{\psi^{ic}}\frac{\partial \psi^c}{\partial r}\right\rangle \mathrm{d}S = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}\sum_{\ell'=0}^{\infty}\sum_{m=-L}^{L}(2\ell+1)\left|H_m^{\ell\ell'}\right|^2 \tag{16}$$

$$H_m^{\ell\ell'} \equiv \sum_{n=-\mathrm{L}}^{\mathrm{L}} h_{mn}^{\ell'\ell} A_n^{\ell'\ell}, \quad L = \min(\ell, \ell') \tag{17}$$

$$h_{mn}^{\ell'\ell} \equiv \sum_{L=|\ell-\ell'|}^{\ell+\ell'} \imath^{-\ell+\ell'}(-1)^{m+n}\left(\ell - m\ell' m|\ell\ell' L0\right)\left(\ell - n\ell' n|\ell\ell' L0\right) \tag{18}$$

Total scattering cross section The total power-flow conservation formula can be written in the form

$$\frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\left[\mathrm{Re}\alpha_\ell + |\alpha_\ell|^2 + \sum_{\ell'=0}^{\infty}\sum_{m=-\mathrm{L}}^{\mathrm{L}}\left|H_m^{\ell\ell'}\right|^2\right] = 0 \tag{19}$$

and the total scattering cross section is given by

$$S = \sigma_c + \sigma_{ic} = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\left[|\alpha_\ell|^2 + \sum_{\ell'=0}^{\infty}\sum_{m=-\mathrm{L}}^{\mathrm{L}}\left|H_m^{\ell\ell'}\right|^2\right] \tag{20}$$

Optical theorem Comparing (14) with (19) the conservation formula can be cast into the form of optical theorem:

$$S = \frac{4\pi}{k}\mathrm{Im}\Phi(0) \tag{21}$$

where $\Phi(0)$ is the forward coherent scattering amplitude. This is the stochastic version of the optical theorem for a random spherical surface.

Angular distribution (differential cross-section) of coherent scattering

$$\sigma_c(\theta)\mathrm{d}S = |\Phi(\theta)|^2\,\mathrm{d}S = \frac{1}{k^2}\left|\sum_{\ell=0}^{\infty}(2\ell+1)\alpha_\ell P_\ell(\cos\theta)\right|^2 \mathrm{d}S \tag{22}$$

where $\mathrm{d}S \equiv \sin\theta\mathrm{d}\theta\mathrm{d}\phi$ denoting the spherical element.

Angular distribution (differential cross-section) of incoherent scattering

$$\sigma_{ic}(\theta)\mathrm{d}S = \frac{1}{k^2}\sum_{\ell'=0}^{\infty}\sum_{m=-\ell'}^{\ell'}\left|\sum_{\ell=0}^{\infty}(2\ell+1)H_m^{\ell\ell'}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}P_\ell(\cos\theta)\right|^2 \mathrm{d}S \tag{23}$$

which depends only on θ as expected. Integrating (22) and (23) gives σ_c and σ_{ic}, respectively.

Numerical calculations are made for the random surface with the Gaussian spectrum

$$|F_\ell|^2 = (\sigma^2/\mathrm{N})\mathrm{e}^{-K^2\ell^2/2}, \quad \ell = 0, 1, 2, \cdots \tag{24}$$

$$\mathrm{N} \equiv \frac{1}{4\pi}\sum_{\ell=0}^{\infty}(2\ell+1)\mathrm{e}^{-K^2\ell^2/2} \tag{25}$$

where $K(<\pi)$ can be roughly considered as the correlation distance (rad).

References

[1] H.Ogura and H.Nakayama, J. Math. Phys. **29**, 851-860 (1988)

[2] Gel'fand,Minlos and Shapiro: Representation of the Rotation and Lorentz Groups and their Applications (Pergamon, London, 1963)

[3] H.Ogura, J. Phys. Soc. Japan,**25**, 586-607 (1968)