Underdetermined DOA Estimation for Uniform Circular Array Based on Sparse Signal Reconstruction

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Abstract - **This paper proposes a novel sparsity-aware method that can estimate more sources than the number of sensors available based on the ℓ1 optimization technique. This approach enforces sparsity by ℓ1 penalization and restricting error by ℓ2- norm which enables the reconstruction of sparse signals. By using the Khatri-Rao (KR) subspace approach, we obtain an increase in the degrees of freedom (DOFs). Thus, using uniform circular array (UCA), we can perform underdetermined DOA estimation for sparse signals. Simulation results confirms the effectiveness of the proposed method.**

Index Terms — **Sparse signal reconstruction; Khatri-Rao; underdetermined DOA estimation; ℓ1-based optimization.**

1. Introduction

The problem of precise localization of multiple sources has received an upsurge of attention recently. In the last few years, there has been a growing interest in algorithms that exploit the sparsity present in various signals and systems for adaptive signal processing [1–4]. The basic idea is to exploit prior knowledge about the sparsity present in the data that need to be processed for applications in system identification, communications and array signal processing [2]. Compressive sensing [3] which is a rapidly expanding area of modern signal processing approximate real life signals by sparse vectors, given some appropriate basis and exploiting the sparse signal structure which reduces signal acquisition cost [1]. By using optimization method described in this paper, accurate signal reconstruction is achieved.

DOA estimation problem in antenna arrays has mainly been confined to uniform linear arrays (ULAs) and uniform circular arrays (UCAs) [5]. Using conventional DOA estimation techniques such as MUSIC and ESPRIT, an M element ULA/UCA can achieve up to (M–1) DOFs [5]. In [6], the KR subspace approach was proposed to increase the DOFs such that underdetermined DOA estimation is possible. In [7], nested linear array was proposed which is capable of performing underdetermined DOA estimation. However, little has been reported for UCAs in terms of their ability to perform underdetermined DOA estimation.

This paper therefore uses the UCA antenna geometry and the KR subspace approach [6] to estimate more sources than the number of antenna elements available. The KR subspace approach increases DOFs of the UCA. Using a sparsity-aware technique with ℓ1 penalization, we are able to reconstruct the sparse signal in an efficient way. Simulation results confirms that the proposed method is capable of performing underdetermined DOA estimation.

2. Signal Model

We consider an M element UCA antenna. We assume that D narrowband sources with wavenumber $k = 2\pi/\lambda$ are impinging on this array from the directions $\phi_1, \phi_2, \cdots, \phi_D$ where, ϕ is the azimuth angle and λ is wavelength. The received signal vector is therefore given by

$$\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{n}(t) \tag{1}$$

where $\boldsymbol{s}(t)$ is a signal vector, and $\boldsymbol{n}(t)$ is noise vector. $\boldsymbol{A} = [\boldsymbol{a}(\theta_1), \boldsymbol{a}(\theta_2), \ldots, \boldsymbol{a}(\theta_D)]$ is the array manifold matrix is given by. We assume that the elevation angle θ is fixed at 90° [5]. The source number D is a *priori* known or accurately estimated. We further assume that the sources are uncorrelated such that the source autocorrelation matrix of $\mathbf{s}(t)$ is diagonal. Thus,

$$\boldsymbol{R}_{xx} = E[\boldsymbol{x}\boldsymbol{x}^H] = \boldsymbol{A}\boldsymbol{R}_{ss}\boldsymbol{A}^H + \sigma_m^2 \boldsymbol{I}_M \tag{2}$$

where $\boldsymbol{R}_{xx}$ is the signal covariance matrix given by the diagonal of signal powers and $\boldsymbol{I}$ is an identity matrix.

3. Proposed Approach

To extend the array aperture, a new array model found using the Khatri-Rao subspace approach. By using the KR approach, we can extend the DOFs for the UCA and be able to perform underdetermined DOA estimation. We consider (3) and derive a new array model. Therefore

$$\begin{aligned} \boldsymbol{y} &= \mathrm{vec}(\boldsymbol{R}_{xx}) = \mathrm{vec}(\boldsymbol{A}\boldsymbol{R}_{ss}\boldsymbol{A}^H) + \mathrm{vec}(\sigma_m^2 \boldsymbol{I}_M) \\ &= (\boldsymbol{A}^* \odot \boldsymbol{A})\boldsymbol{p} + \sigma_m^2 \boldsymbol{I}_M \end{aligned} \tag{3}$$

where $\boldsymbol{p}$ is equivalent to source signal vector whilst $\sigma_m^2 \boldsymbol{I}_M$ is noise and $\boldsymbol{y}$ is the array's received signal whose manifold is given by $\boldsymbol{B} = (\boldsymbol{A}^* \odot \boldsymbol{A})$. The steering matrix of array with virtual elements will be given by $\boldsymbol{B} = [\boldsymbol{b}(\phi_1), \boldsymbol{b}(\phi_2), \ldots, \boldsymbol{b}(\phi_D)]^T$ which is $M^2 \times D$ matrix.

Thus, instead of using (1), we can apply the problem of DOA estimation to (3). We therefore consider (3) as a sparse signal representation problem given by

$$\boldsymbol{y} = \boldsymbol{B}\boldsymbol{p} + \sigma_m^2 \boldsymbol{I}_M \quad (4)$$

To extend ℓ_1 penalization to (4), we need to appropriately choose the optimization criteria which is $\min\|\boldsymbol{p}\|_1$ subject to $\|\boldsymbol{y}-\boldsymbol{B}\boldsymbol{p}\|_2^2 \le \beta^2$, where β is a parameter specifying how much noise we wish to allow. Therefore, an unconstrained form of this objective function is

$$\min\|\boldsymbol{y}-\boldsymbol{B}\boldsymbol{p}\|_2^2 + \lambda\|p\|_1 \quad (5)$$

The ℓ_2 term in (5) forces the residual $\boldsymbol{y}-\boldsymbol{B}\boldsymbol{p}$ to be small and λ controls the tradeoff between the sparsity of the spectrum and residual norm [4]. In a practical setting, $\boldsymbol{y}$ in (4) can be estimated from N snapshots such that $\Delta\boldsymbol{y} = \hat{\boldsymbol{y}} - \boldsymbol{y}$. Let $\boldsymbol{W}$ be a weighting matrix given by $\boldsymbol{W} = (1/N)\boldsymbol{R}_{xx}^T \otimes \boldsymbol{R}_{xx}$. Let $\hat{\boldsymbol{p}}$ be the estimate of $\boldsymbol{p}$, the DOA estimation problem is therefore given by the following ℓ_1-norm minimization

$$\min_{\hat{\boldsymbol{p}}}\|\hat{\boldsymbol{p}}\|_1 \text{ subject to} \|\hat{\boldsymbol{y}}-\boldsymbol{B}\hat{\boldsymbol{p}}\|_2^2 + \lambda\|\hat{\boldsymbol{p}}\|_1 \quad (6)$$

By introducing two parameters $\boldsymbol{W}$ and $\beta = \sqrt{\chi^2(M^2)}$, the DOA estimation can thus be reduced to

$$\min_{\hat{\boldsymbol{p}}}\|\hat{\boldsymbol{p}}\|_1 \text{ subject to} \left\|\boldsymbol{W}^{(-1/2)}[\hat{\boldsymbol{y}}-\boldsymbol{B}\hat{\boldsymbol{p}}]\right\|_2^2 + \beta^2 \quad (7)$$

The problem (7) is a second-order cone program problem solved efficiently by optimization software such as CVX.

4. Numerical Results

In order to evaluate the performance of the proposed sparse signal reconstruction technique for underdetermined DOA estimation, simulations were conducted. We examine the 6 element UCA (M = 6) with 7 narrowband sources (D = 7) for signals impinging from the directions ϕ = [15°, 36°, 70°, 90°, 112°, 130°, 145°]. The radius of the UCA is r = λ, for 1000 trials. We examine the performance of an extension of the KR-MUSIC [6], and ℓ_1-based optimization technique with the Cramer-Rao lower bound (CRLB).

Fig. 1 shows the RMSE as a function of SNR for MUSIC and ℓ1-based optimization for underdetermined DOA estimation. The ℓ1-based optimization method has good performance compared to MUSIC. Fig. 2 shows the RMSE as a function of number of snapshots for MUSIC and ℓ_1-based optimization in an underdetermined DOA estimation case. The ℓ_1-based optimization method which assumes sparse signals has good performance compared to MUSIC as well but becomes stagnant even though the number of snapshots increases.

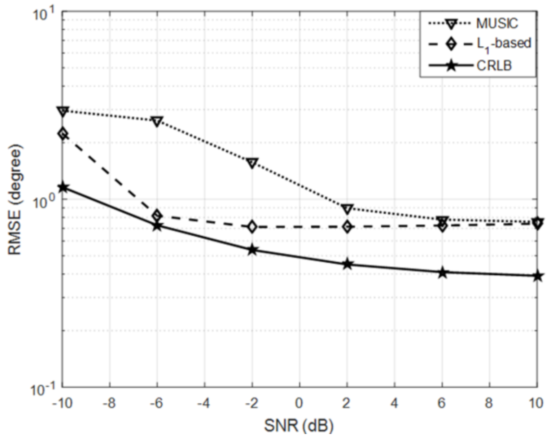

Fig. 1. RMSE versus SNR using UCA with M = 6, snapshots = 10000.

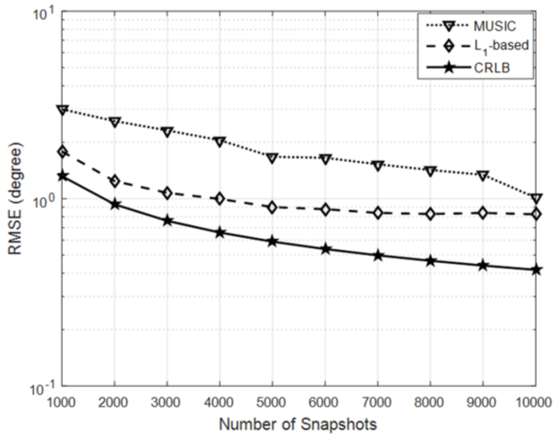

Fig. 2. RMSE versus snapshots using UCA with M = 6, SNR = 5 dB.

5. Conclusion

In this paper, underdetermined DOA estimation problem for UCA antenna based on sparse signal reconstruction has been discussed. An increase in the DOFs is obtained by the KR subspace approach which enables underdetermined DOA estimation. By using the ℓ1-based optimization method, we have confirmed that this array can estimate more sources than the number of sensors available. Simulation results has confirmed the performance of the proposed method.

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