SOLUTION METHOD OF WEBER FUNCTION OF SPECIAL COORDINATE SYSTEM IN ELECTRIC FIELD

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Abstract : In this paper ,a special coordinate system—parabolic cylindrical surface coordinate system and a special function—Weber function's applications in electric field are presented .The problem of electric field calculation with parabolic cylindrical boundary surface is solved simply and systematically. Now we know well the calculation of electric field whose boundary shape is cubic surface, spherical surface, spherical surface, or elliptical cylindrical surface. So far, calculation of whose boundary shape is parabolic cylindrical has not been presented. The paper solved the problem better. First, a math model using Weber function is given. Then, the author gives an illustrate example of parabolic cylindrical surface to explain the model.

Key words: Special coordinate system, Special function, parabolic cylindrical surface, and Weber functions.

1. INTRODUCTION

In the calculation problem of electric field , when the boundary conditions is special figure, its strict math analytic present. With the development of electric industry and the presence of several special antennae and communication equipment, the calculation of electric field with common boundary shape has been out of time. So we must search some new math models and calculation methods of electric field with special boundary.

The old methods to solve electric field with parabolic cylindrical surface boundary are following as 1test method;2) numerical analysis method;3)strict analytic method. The test method has three shortcomings. First of all, it is limited by test equipment which is expensive .Generally, common firms can not afford. Next, sometimes due to the limits of test, we can not test the electric field at all. Last, even if we satisfy the two conditions above, test results can hardly be accurate due to errors of the method and effects of earth surface's landform. The numerical analysis method has the same shortcomings. And it needs a huge computer. In addition, it can not get accurate results. At the same time, it is quite complicated and need programming. But strict analytic method does not need any equipment. And problems are solved by using math model, Weber equation and Weber function. Of course, its results are accurate.

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2. MATH MODEL

As separation of variables method is always used in parabolic cylindrical surface coordinate system so we first, introduce the math tool briefly.

2.1 Laplace equation and parabolic cylindrical coordinate system [1]

Coordinate surface is specified as:

$$y^2 = \beta^2(\beta^2 - 2x)$$

$$y^2 = \alpha^2(\alpha^2 + 2x)$$

If β is a constant, the coordinate surface is parabolic cylindrical. If α is a constant, the coordinate surface is parabolic cylindrical too. If is a constant, the coordinate surface is a plane.

Laplace equation is defined as:

$$\frac{1}{\beta^2+\alpha^2}\left(\frac{\partial^2\varphi}{\partial\beta^2}+\frac{\partial^2\varphi}{\partial\alpha^2}\right)+\frac{\partial^2\varphi}{\partial z^2}=0$$

2.2 Laplace equation's series solution. Let

$$\varphi = \mathrm{I}(\beta)E(\alpha)Z(z) \qquad (1)$$

Where $\mathrm{I}(\beta), E(\alpha)$, and Z (z) are respectively the function of β,α , and z. Taking (1) into Laplace equation, we obtain the following equation:

$$\frac{1}{\beta^2+\alpha^2}\left(\frac{1}{I}\frac{d^2I}{d^2\beta}+\frac{1}{E}\frac{d^2E}{d\ \alpha^2}\right)+$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2}=0 \qquad (2)$$

If the above three units' sum is zero, it must satisfy:

$$\frac{1}{Z}\frac{d^2Z}{dz^2}=\mp m^2$$

So can be a constant. In this case, we can set obtain

$Z = A\cos(mz) + B\sin(mz)$

(when m^2 is negative)

$Z = Ach(mz) + Bsh(mz)$

(when m^2 is positive)

Where A and B are constants. Taking

$\frac{1}{Z}\frac{d^2Z}{dz^2}=\mp m^2$ into (2),

we can get the following expression:

$$\frac{1}{\beta^2+\alpha^2}\left(\frac{1}{I}\frac{d^2I}{d^2\beta}+\frac{1}{E}\frac{d^2E}{d\ \alpha^2}\right)+$$

$$\mp m^2=0 \qquad (3)$$

Multiplying (3) by ($\beta^2+\alpha^2$), (3) can be rewritten as follows:

$$\frac{1}{I}\frac{d^2I}{d\beta^2}+\frac{1}{E}\frac{d^2E}{d\alpha^2}+m^2(\beta^2+\alpha^2)=0 \qquad (4)$$

$$\frac{1}{I}\frac{d^2I}{d\beta^2}+\frac{1}{E}\frac{d^2E}{d\alpha^2}-m^2(\beta^2+\alpha^2)=0 \qquad (5)$$

Let's discuss (4). If we want the left side of (4) to be zero, (4) are satisfied as the following:

$$\begin{cases}\frac{1}{I}\frac{d^2I}{d\beta^2}+m^2\beta^2=q^2\\ \frac{1}{E}\frac{d^2E}{d\alpha^2}+m^2\alpha^2=-q^2\\ \frac{d^2I}{d\beta^2}+(m^2\beta^2-q^2)I=0\\ \frac{d^2E}{d\alpha^2}+(m^2\alpha^2+q^2)E=0\end{cases} \qquad (6)$$

or

$$\begin{cases}\frac{1}{I}\frac{d^2I}{d\beta^2}+m^2\beta^2=-q^2\\ \frac{1}{E}\frac{d^2E}{d\alpha^2}+m^2\alpha^2=q^2\\ \frac{d^2I}{d\beta^2}+(m^2\beta^2+q^2)I=0\\ \frac{d^2E}{d\alpha^2}+(m^2\alpha^2-q^2)E=0\end{cases} \qquad (7)$$

Next, let's discuss (5). If we want the left of (5) to be zero, (5) must satisfy the following several equations.

$$\begin{cases}\frac{1}{I}\frac{d^2I}{d\beta^2}+m^2\beta^2=q^2\\ \frac{1}{E}\frac{d^2E}{d\alpha^2}+m^2\alpha^2=q^2\\ \frac{d^2I}{d\beta^2}-(m^2\beta^2+q^2)I=0\\ \frac{d^2E}{d\alpha^2}-(m^2\alpha^2-q^2)E=0\end{cases} \qquad (8)$$

or

$$\begin{cases}\frac{1}{I}\frac{d^2I}{d\beta^2}-m^2\beta^2=-q^2\\ \frac{1}{E}\frac{d^2E}{d\alpha^2}-m^2\alpha^2=+q^2\\ \frac{d^2I}{d\beta^2}-(m^2\beta^2-q^2)I=0\\ \frac{d^2E}{d\alpha^2}-(m^2\alpha^2+q^2)E=0\end{cases} \qquad (9)$$

Where (6), (7), (8), and (9) are Weber equations and their solutions are Weber function. The standard Weber equations and Weber functions [1] are specified as :

$$\begin{cases}\frac{d^2y}{dx^2}-[R^2(p+1/2)+R^4x^4/4]y=0\\ y=AW_e(p,jRx)+BW_0(p,jRx);\end{cases}$$

(10)

$$\begin{cases}\frac{d^2y}{dx^2}+[R^2(p+1/2)-R^4x^4/4]y=0\\ y=AW_e(p,Rx)+BW_0(p,Rx);\end{cases}$$

(11)

Where A and B are parameters , $W_e(p,R_x)$, and $W_0(p,R_x)$ are Weber functions. Their forms follow as:

$$W_e(p, jRx) = e^{\frac{(Rx)^2}{4}} \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{(2n)!} p(p-2) \right.$$
$$\left. (p-4)...[p-2(n-1)](Rx)^{2n} \right\};$$
$$W_0(p, jRx) = e^{\frac{(Rx)^2}{4}} (jRx) \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} \right.$$
$$\left. (p-1)(p-3)...[p-(2n-1)](Rx)^{2n} \right\};$$
$$W_e(p, Rx) = e^{-\frac{(Rx)^2}{4}} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} p(p-2) \right.$$
$$\left. (p-4)...[p-2(n-1)](Rx)^{2n} \right\};$$
$$W_o(p, Rx) = e^{-\frac{(Rx)^2}{4}} (Rx) \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \right.$$
$$\left. (p-1)(p-3)...[p-(2n-1)](Rx)^{2n} \right\};$$

3. ILLUSTRATIVE EXAMP IN ENGINEERING

As a direct application to the theory, a parabolic cylindrical surface is considered, The parabolic cylindrical surface is of height b. When α is a constant, α_0, φ is a constant, V. The top and bottom surfaces' electric potentials are all zero. What is the electric potential of any point the parabolic cylindrical surface?

The above example is relative to z and α, but is not relative to β. So it is a two-dimension problem. The Laplace equation can be written as following:

$$\frac{1}{\alpha^2}\left(\frac{\partial^2 \varphi}{\partial \alpha^2}\right) + \frac{\partial^2 \varphi}{\partial z^2} = 0 \qquad (12)$$

Let $\varphi = E(\alpha)Z(z)$ (13)

Taking (13) into (12), we can obtain:

$$\frac{1}{\alpha^2}\frac{d^2 E}{d\alpha^2} Z + \frac{d^2 Z}{dz^2} E = 0 \qquad (14)$$

Dividing (14) by EZ, we get

$$\frac{1}{\alpha^2 E}\frac{d^2 E}{d\alpha^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0 \qquad (15)$$

If we want the right of (15) is zero, (15) is satisfied as the following:

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = \mp m^2$$

But only when $\frac{1}{Z}\frac{d^2 Z}{dz^2} = -m^2$,

Can we satisfy the boundary conditions .. From $\frac{1}{Z}\frac{d^2 Z}{dz^2} = -m^2$, we can obtain:

$$\begin{cases} Z = A_n \sin(\frac{n\pi}{b} z) \\ m^2 = \frac{n^2\pi^2}{b^2} \end{cases}$$

Taking $\frac{1}{Z}\frac{d^2 Z}{dz^2} = -\frac{n^2\pi^2}{b^2}$ into (15),we can get

$$\frac{d^2 E}{d\alpha^2} - \alpha^2 E\left(\frac{n^2\pi^2}{b^2}\right) = 0 \qquad (16)$$

Comparing (16) with the standard Weber equation, we can obtain:

$$R^4 = \frac{4n^2\pi^2}{b^2} \quad \text{and} \quad p = -1/2$$

So (16) can be rewritten using the standard Weber equation's form as follow:

$$\frac{d^2 E}{d\alpha^2} + \left[\frac{2n\pi}{b}\left(-\frac{1}{2} + \frac{1}{2}\right) - \frac{4n^2\pi^2}{4b^2} a^2\right] E = 0 \qquad (17)$$

Then, the solution of (17) is following as:

$$E - AW_e(p, R\alpha) + BW_0(p, R\alpha) \qquad (18)$$

because when α is zero, $W_0(p, R\alpha)$ is zero. So we do not take care of $W_0(p, R\alpha)$. Then (18) can be rewritten as:

$$E = AW_e\left[-\frac{1}{2}, \sqrt{\frac{2n\pi}{b}}\alpha\right] \qquad (19)$$

Taking (19) and Z= $A_n \sin(\frac{n\pi}{b} z)$ into (13), we can obtain the following solution:

$$\varphi = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{b} z\right) W_e\left[-\frac{1}{2}, \sqrt{\frac{2n\pi}{b}}\alpha\right] \qquad (20)$$

From the conditions of the example, we know when $\alpha = \alpha_0$, we can get

$$V = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{b} z\right) W_e\left[-\frac{1}{2}, \sqrt{\frac{2n\pi}{b}}\alpha_0\right] \qquad (21)$$

To decide parameter A_n, we must use he property of orthogonality of trigonometric function. We can et the following expression from (21):

$$\int_0^b V \sin(\frac{n\pi}{b}z)dz = \int_0^b \sum_{n=1}^{\infty} A_n \sin^2(\frac{n\pi}{b}z)$$

$$W_e[-\frac{1}{2},\sqrt{\frac{2n\pi}{b}}\alpha_0]dz$$

$$V\frac{b}{n\pi}(-\cos\frac{n\pi}{b})\Big|_0^b = \frac{b}{2}A_n W_e[-\frac{1}{2},\sqrt{\frac{2n\pi}{b}}\alpha_0]$$

So ,from the boundary conditions of the example, we can obtain the expression of A_n as follows:

$$A_n = \frac{4V}{n\pi W_e[-\frac{1}{2},\frac{2n\pi}{b}\alpha_0]} \qquad (22)$$

Where n=1,3,5,7…n is an odd number. Taking (22) into (20),we can et the last solution:

$$\varphi = \sum_{n=1}^{\infty} \frac{4V}{n\pi W_e[-\frac{1}{2},\frac{2n\pi\alpha_0}{b}]}$$

$$\sin(\frac{n\pi}{b}z)W_e[-\frac{1}{2},\frac{2n\pi}{b}\alpha]$$

Where n=1,3,5,7,… From the above analysis , we get the solution of special boundary conditions in parabolic cylindrical surface. If the boundary conditions change into: when z is zero, φ is V_1 and when z is b, φ is V_2, and in this case, boundaries are all the non-homogenous boundary conditions. We must solve it with method of superposition. Then form of solution using method of superposition follows as: $\varphi = \varphi_1 + \varphi_2 + \varphi_3$, where φ_1 is the solution in the case that when α_0, φ is V, when z is zero, φ is zero, and z is b, φ is zero. Then form of φ is presented in the above analysis; φ_2 is the solution in the case that when α is α_0, φ is zero, when z is zero, φ is V_1, and when z is b, φ is zero; φ_3 is the solution in the case that when $\alpha = \alpha_0$, φ is zero , when z is b, φ is zero, and when z is b, φ is V_2.

In the case , is we analyze φ_2 and φ_3, the functions of z direction are not sinusoidal functions but hyperbolic functions. In addition , when we solve φ_2 and φ_3, we should use the property of orthogonality of Weber function to decide the coefficient A.

4. CONCLUSION

The application of special coordinate system—parabolic cylindrical surface coordinate system and special function—Weber function in electric field is presented. So far, in engineering electromagnetic field, we have known four coordinate systems. The method of the paper builds the 5th coordinate systems, parabolic cylindrical surface. Its simplicity and accuracy characterize the method. It is concluded that with the use o Weber functions and property of orthogonality of unction, the electric potential in the coordinate system can be obtained. A parabolic cylindrical surface's electric potential was considered for illustrating the general procedure and result for this problem was also given.

REFERENCE

[1] Ren rang, "Theory of antennae", China people posts and telecom publication , pp. 460-462,Dec. 1980. China

[2] Wangqinbin , Liuping" The Finite Element Solution Method of Electromagnetics" Chongqing Univ.Dec.1996.China.