ITERATIVE TECHNIQUE TO CORRECT PROBE POSITION ERRORS IN PLANAR NEAR-FIELD TO FAR-FIELD TRANSFORMATIONS

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We have developed a general theoretical procedure to take into account probe position errors when planar near-field data are transformed to the far field. If the probe position errors are known, we can represent the measured data as a Taylor series, whose terms contain the error function and the ideal spectrum of the antenna. Then we can solve for the ideal spectrum in terms of the measured data and the measured position errors by inverting the Taylor series. This is complicated by the fact that the derivatives of the ideal data are unknown; that is, they can only be approximated by the derivatives of the measured data. This introduces additional computational errors, which must be properly taken into account. We have shown that the first few terms of the inversion can be easily obtained by simple approximation techniques, where the order of the approximation is easily specified. A more general solution can also be written by formulating the problem as an integral equation and using the method of successive approximations to obtain a general solution. An important criterion that emerges from the condition of convergence of the solution to the integral equation is that the total averaged position error must be less than some fraction of the Nyquist rate for the antenna under test.

1. Analytical Error Expressions

The field radiated by an antenna can be described as the superposition of an infinite number of plane waves whose wavenumbers k are constant [1]. We can write that $\vec{k} = (\vec{K}, \gamma)$, where $\vec{k} \cdot \vec{k}$ =constant, $\vec{K} = (k_x, k_y)$, and $\gamma^2 = k^2 - K^2$ gives the magnitude of the z component of the propagation vector. The received near-field signal b_0' measured by a probe whose receiving coefficients are $S_{02}'(\vec{K})$ is

$$b_0'(x, y, z) = F' a_0 \iint \vec{T}_{10}(\vec{K}) \cdot \vec{S}_{02}'(\vec{K})\, e^{i\gamma z}\, e^{i\vec{K}\cdot\vec{P}} dk_x\, dk_y \tag{1}$$

where $F' = 1/(1 - \Gamma_l \Gamma_\rho)$, Γ_l and Γ_ρ are reflection coefficients for the load and probe, respectively; $T_{10}(\vec{K})$ are the transmission coefficients of the antenna under test; a_0 is the amplitude of the incident wave produced by the generator at the terminal surface S_0; z is the distance of the near-field scan plane from S_1, which is a plane situated in front of the antenna defining $z = 0$, and the position vector $\vec{P} = (x_1\hat{x}, x_2\hat{y})$, where $\hat{x}$ and $\hat{y}$ are unit vectors. Equation (1) assumes that multiple reflections are negligible; the presence of multiple reflections in a real measurement range is minimized by judiciously choosing the position of the plane of measurement and the size and design of the probe.

Since eq (1) is a Fourier transform, the quantity $D(\vec{K}) \equiv 4\pi^2 a_0 F' \vec{T}_{10}(\vec{K}) \cdot \vec{S}_{02}'(\vec{K})$ can be immediately written in terms of near field data. Thus,

$$D(\vec{K}) = \iint b_0'(\vec{P}, z) e^{-i\vec{k}\cdot\vec{x}} dx\, dy, \tag{2}$$

where $\vec{x} = (x_1, x_2, x_3) = \vec{P} + x_3\hat{z}$.

Since the z dependence of the near-field quantity b_0' in eq (1) appears only in the exponential, we can immediately write that

$$\frac{\partial^n b_0'}{\partial z^n} = \frac{i^n}{4\pi^2} \iint D(\vec{K})\, \gamma^n\, e^{i\gamma z} e^{i\vec{K}\cdot\vec{P}}\, dk_1\, dk_2. \tag{3}$$

Similarly, the partial derivatives with respect to x_j for $j = 1, 2$ are given by

$$\frac{\partial^n b_0'}{\partial x_j^n} = \frac{i^n}{4\pi^2} \iint D(\vec{K})\, k_j^n\, e^{i\gamma z} e^{i\vec{K}\cdot\vec{P}}\, dk_1\, dk_2. \tag{4}$$

If these expressions can be evaluated, then first-order corrections can be introduced into the data. We assume that the probe's position is known accurately and is given by

$$\vec{x} + \delta\vec{x}(\vec{x}), \tag{5}$$

where $\vec{x}$ is the position of the probe on an ideal near-field range, where measurements are made on a regularly spaced (x_1, x_2) grid, and $\delta\vec{x}(\vec{x})$ is the deviation in the probe's position from the ideal grid. A thorough discussion of the effects of such displacement errors on the far-field pattern has been presented in [2,3]. Some of the basic considerations relevant to the current subject are included here.

The near-field quantity $b_0'(\vec{x} + \delta\vec{x}(\vec{x}))$ is measured at the locations given by eq (5). However, this function is assumed to be defined on the regular grid $\vec{x}$ when the spectrum is obtained numerically using Fourier techniques. We can write the Taylor expansion at $\vec{x}$,

$$b_0'(\vec{x} + \delta\vec{x}(\vec{x})) \equiv b_0'(\vec{x}) + \frac{\partial b_0'(\vec{x})}{\partial x_j}\delta x_j + \frac{1}{2}\frac{\partial^2 b_0'(\vec{x})}{\partial x_i \partial x_j}\delta x_i \delta x_j + \cdots, \tag{6}$$

which defines the measured data on the left in terms of the unknown field quantities on the right. Also, the error-contaminated spectrum $D_e(\vec{K})$ can be expressed in terms of the measured data as (see eq (2))

$$D_e(\vec{K}) = \iint b_0'(\vec{x} + \delta\vec{x}(\vec{x})) e^{-i\vec{k}\cdot\vec{x}}\, dx\, dy. \tag{7}$$

We can differentiate eq (6) with respect to x_ℓ, $\ell = 1, 2, 3$, so

$$\frac{\partial}{\partial x_\ell} b_0'(\vec{x} + \delta\vec{x}(\vec{x})) = \frac{\partial b_0'(\vec{x})}{\partial x_\ell} + \frac{\partial}{\partial x_\ell}\left(\frac{\partial b_0'(\vec{x})}{\partial x_j}\delta x_j\right) + \cdots. \tag{8}$$

Equation (8) can be differentiated again, with the result that

$$\frac{\partial^2}{\partial x_i \partial x_j} b_0'(\vec{x} + \delta\vec{x}(\vec{x})) = \frac{\partial^2 b_0'(\vec{x})}{\partial x_i \partial x_j} + O(\delta x). \tag{9}$$

Equation (7) now yields

$$D_e(\vec{K}) = D(\vec{K}) + \iint [\frac{\partial b_0'(\vec{x}+\delta\vec{x})}{\partial x_j}\delta x_j + \frac{1}{2}\frac{\partial^2 b_0'(\vec{x}+\delta\vec{x})}{\partial x_i \partial x_j}\delta x_i \delta x_j - \frac{\partial}{\partial x_\ell}(\frac{\partial b_0'(\vec{x}+\delta\vec{x})}{\partial x_j}\delta x_j)\delta x_\ell + O(\delta x_i \delta x_j \delta x_\ell)]e^{-i\vec{k}\cdot\vec{x}}\, dx\, dy. \tag{10}$$

which is a second-order correction to $D_e(\vec{K})$.

2. The True Spectrum as a Solution to an Inhomogeneous Integral Equation

There is a general procedure for writing the nth-order approximation to $D(\vec{K})$. Equations (1), (2) and (7) can be combined to yield

$$D(\vec{K}) = D_e(\vec{K}) + \frac{1}{4\pi^2}\iint D(\vec{K}')\iint (1 - e^{i\vec{k}'\cdot\delta\vec{x}(\vec{x})})e^{i(\vec{k}'-\vec{k})\cdot\vec{x}}\, dx\, dy\, dk_x'\, dk_y'. \tag{11}$$

Equation (11) is of the form of the Fredholm integral equation [4], which in two dimensions is written as

$$f(x,y) = g(x,y) + \int_a^b M(x,y;x',y')f(x',y')\, dx'\, dy'. \tag{12}$$

Symbolically this is $f = g + Mf$, where M is an operator.

A solution of the general integral equation (12) can be obtained by the method of successive approximation (Neumann series) [4]. Symbolically, the nth-order solution is given by

$$f_n = g + Mf_{n-1} = (1 + M + M^2 + \ldots + M^{n-1})g. \tag{13}$$

This solution is unique if the series in eq (13) converges uniformly [5]. The condition for convergence is that the integral operator M be bounded so that its least upper bound or norm, $\|M\|$, is less than 1. An alternate condition is that the product of the range of integration and of the maximum value of the kernel is less than 1 [5]. An estimate of the norm is given by

$$\|M\|^2 < \iint |M(x,y;x',y')|^2\, dx\, dy\, dx'\, dy', \tag{14}$$

and the kernel is, in our example,

$$M(\vec{K},\vec{K}') = \frac{1}{4\pi^2}\iint (1 - e^{i\vec{k}'\cdot\delta\vec{x}(\vec{x})})e^{i(\vec{k}'-\vec{k})\cdot\vec{x}}\, dx\, dy. \tag{15}$$

The maximum value of the kernel occurs at $k = k'$, where its first-order approximation can be written as

$$|M^{(1)}(\vec{K},\vec{K})| = \frac{1}{4\pi^2}|\iint \vec{k}\cdot\delta\vec{x}\, dx\, dy|. \tag{16}$$

Thus,

$$|M^{(1)}(\vec{K},\vec{K})| \leq \frac{k}{4\pi^2}|\iint \delta\vec{x}\, dx\, dy| = \frac{kA}{4\pi^2}[\delta\vec{x}] \tag{17}$$

where $[\delta\vec{x}]$ is the average of the error displacement function in the scan plane. Since the range of integration is $\Delta k = 2k_i$, where $k_i \approx 2\pi/\lambda$, the second condition of convergence stated above can be written, to first order, as

$$\frac{[\delta\vec{x}]}{(\lambda/2)} < \frac{1}{\pi}\frac{(\lambda/2)^2}{A}. \tag{18}$$

If the near field is sampled at $\lambda/2$ intervals, then the ratio on the right side above is essentially the inverse of the number of measurement points N. Then the inequality (18) becomes

$$N[\delta\vec{x}] < \frac{1}{\pi}(\lambda/2) \tag{19}$$

which states that the *total averaged* displacement error has to be less than a fraction of the grid spacing $\lambda/2$. This is a rather stringent condition on the acceptable size of displacement errors in a near-field measurement range, which must be satisfied if we want to recover the true spectrum from error-contaminated near-field data. The condition essentially means that the displacement errors must be small enough so that the sampling criterion [6] is not violated in an average sense. A similar expression can be derived for second-order errors, which is $N[\delta\vec{x}\cdot\delta\vec{x}] < \frac{2}{\pi^2}(\lambda/2)^2$.

To facilitate evaluation of the integrals in eq (11), we expand the exponential term containing $\delta\vec{x}$ to second order, and write the second-order iterative solution of eq (11) as (using the abbreviations $d\vec{x} \equiv dx\,dy$ and $d\vec{K} \equiv dk_1\,dk_2$)

$$\begin{aligned} D(\vec{K}) = D_e(\vec{K}) &- \frac{i}{4\pi^2}\int\!\!\int \delta x_j e^{-i\vec{k}\cdot\vec{x}}\{\int\!\!\int k'_j D_e(\vec{K}')e^{i\vec{k}'\cdot\vec{x}}d\vec{K}'\}\,d\vec{x} \\ &+\frac{1}{8\pi^2}\int\!\!\int \delta x_j\delta x_\ell e^{-i\vec{k}\cdot\vec{x}}\{\int\!\!\int k'_j k'_\ell D_e(\vec{K}')e^{i\vec{k}'\cdot\vec{x}}d\vec{K}'\}\,d\vec{x} \\ -\frac{1}{16\pi^4}\int\!\!\int d\vec{x}\,\Big\{\delta x_j e^{-i\vec{k}\cdot\vec{x}}&\int\!\!\int d\vec{K}'\Big[k'_j e^{i\vec{k}'\cdot\vec{x}}\int\!\!\int d\vec{x}\,'\Big(\delta x_\ell e^{-i\vec{k}'\cdot\vec{x}'}\int\!\!\int k''_\ell D_e(\vec{K}'')e^{i\vec{k}''\cdot\vec{x}'}d\vec{K}''\Big)\Big]\Big\}. \end{aligned} \tag{20}$$

Higher-order iterations can be readily obtained, but we will not do so here. Each iterated integral above is a Fourier transform; hence, these integrals can be evaluated efficiently using FFT codes.

References

[1] Kerns, D. M. Plane wave scattering-matrix theory of antennas and antenna-antenna interactions. Nat. Bur. Stand. (U.S.) Monogr. 162; 1981.

[2] Newell, A. C. Error analysis techniques for planar near-field measurements. IEEE Trans. Antennas and Propagat., 36, 754-768; 1988 June.

[3] Muth, L. A. Displacement errors in antenna near-field measurements and their effect on the far field. Nat. Bur. Stand. (U.S.) Tech.

[4] Stakgold, I. Boundary value problems of mathematical physics, v. 1, MacMillan Series in Advanced Mathematical and Theoretical Physics, London, The MacMillan Company, Collar-MacMillan Limited; 1967. 340 p.

[5] Riesz, F.; Sz.-Nagy, B. Functiomal Analysis, New York, Frederick Ungar Publishing Co.; 1955. 468 p.

[6] Landu, H. J. Sampling, data transmission, and the Nyquist rate, IEEE Trans. Antennas and Propagat., 55, 1701-1706; 1967 October.