Path Integrals and Wedge Diffraction

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1 Introduction

Path integrals[1][2] are known for the third way of quantization procudures in quantum mechanics. Path integrals are suited for formal theoretical developments in quantum field theory, and most of the studies were concentrated in that area. Recently, some activities appeared which apply the path integrals to the exact solutions for potential scattering or boundary value problems of the Schrödinger equation. Some canonical quantization solutions have been reconstructed by the path integrals[3]. In this paper, an exact path integral solution is constructed for the wedge diffraction problem of the scalar Helmholtz equation, which employs mathematical equivalence between the Helmholtz equation and the Schrödinger equation for a steady-state free scalar particle.

2 Schrödinger and Helmholtz Equations

The Feynman propagator $K(q'', q', E)$ for a steady-state free scalar particle of the mass m at the energy E is defined as follow:

$$\left(\nabla^2 + \frac{2mE}{\hbar^2}\right) K(q'', q', E) = \delta(q'' - q'), \tag{1}$$

where q'', q' are three dimensional coordinates and $\hbar$ is defined by the division of Planck's constant by 2π. The Green's function $G(q'', q', k)$ of free space scalar Helmholtz equation at the wave number k satisfies the following equation:

$$\left(\nabla^2 + k^2\right) G(q'', q', k) = -\delta(q'' - q'). \tag{2}$$

If the conditions $2m/\hbar^2 = 1, E = k^2, G(q'', q', k) = -K(q'', q', E)$ are posed. (1) and (2) become mathematically equivalent. The relation between the time dependent Feynman propagator $K(q'', q', T)$ and $G(q'', q', k)$ is given as follow:

$$G(q'', q', k) = i\int_0^\infty dT K(q'', q', T) e^{ik^2 T} \tag{3}$$
$$(\hbar = 1, \quad m = 1/2).$$

By means of the above relations, solutions of the Helmholtz equation can be constructed through the path integrals.

3 Feynman Propagator for a Wedge

Consider the wedge diffraction problem in Fig.1 for the scalar Helmholtz equation. Equivalent quantum mechanical problem is "determination of the motion of free scalar particle of the mass 1/2 around the wedge shaped infinite potential wall $V(\phi)$." The Hamiltonian path integral[2] for this problem is as follow:

$$K(q'',q',T) = \int Dq(t)Dp(t)\exp\left[i\int_0^T dt\left\{p\cdot\dot{q} - p^2 - V(\phi)\right\}\right], \tag{4}$$

where $\dot{x} = dx/dt$, and the path integration is defined as the following continuum limit:

$$\begin{aligned}\int Dq(t)Dp(t) \;&\equiv\; \lim_{N\to\infty}\frac{1}{(2\pi\hbar)^{3N}}\int dx_1\cdots dx_{N-1}dy_1\cdots dy_{N-1}dz_1\cdots dz_{N-1} \\ &\quad\cdot\int dp_{x_1}\cdots dp_{x_N}dp_{y_1}\cdots dp_{y_N}dp_{z_1}\cdots dp_{z_N}.\end{aligned} \tag{5}$$

The following separation is possible:

$$K(q'',q',T) = K_\rho(\rho'',\rho',T)K_z(z'',z',T). \tag{6}$$

As the system is uniform in the z direction, K_z becomes one dimensional free particle propagator:

$$\begin{aligned}K_z(z'',z',T) &= \int Dz(t)Dp_z(t)\exp\left\{i\int_0^T dt\left(p_z\dot{z} - p_z^2\right)\right\} \\ &= \left(\frac{1}{4\pi Ti}\right)^{1/2}\exp\left\{\frac{1}{4T}(z''-z')^2\right\}.\end{aligned} \tag{7}$$

The propagator in the transverse direction $K_\rho(\rho'',\rho',T)$ is defined as follow:

$$\begin{aligned}K_\rho(\rho'',\rho',T) &= \int Dx(t)Dy(t)Dp_x(t)Dp_y(t) \\ &\quad\cdot\exp\left[i\int_0^T dt\left\{p_x\dot{x} + p_y\dot{y} - (p_x^2 + p_y^2) - V(\phi)\right\}\right].\end{aligned} \tag{8}$$

After transforming K_ρ into the cylindrical coordinate[4], the angular dependent part reduces to one dimensional free space propagator in the wedge shaped infinite potential well, then the method of images is applied for the solution. The radial part can be integrated through the following formula;

$$\begin{aligned}&\int_{\rho'}^{\rho''} D\rho(t)\exp\left\{i\int_0^T\left(\frac{\dot{\rho}^2}{4} - \frac{\nu^2 - 1/4}{\rho^2}\right)\right\} \\ &= -i(\rho''\rho')^{1/2}\frac{1}{2T}\exp\left\{\frac{i}{4T}(\rho''^2 + \rho'^2)\right\}I_\nu\left(\frac{\rho''\rho'}{2Ti}\right).\end{aligned} \tag{9}$$

where $I_\nu(z)$ is a modified Bessel function.
The resultant expression for the transverse propagator is as follow:

$$\begin{aligned}&K_\rho(\rho'',\rho',T) \\ &= \frac{1}{\alpha Ti}\sum_{l=1}^{\infty}\sin\left(\frac{l\pi\phi''}{\alpha}\right)\sin\left(\frac{l\pi\phi'}{\alpha}\right)\exp\left\{\frac{i}{4T}(\rho''^2 + \rho'^2)\right\}I_{l\pi/\alpha}\left(\frac{\rho''\rho'}{2Ti}\right).\end{aligned} \tag{10}$$

4 Two Dimensional Wedge Diffraction

The Green's function $G_\rho(\rho'', \rho', k)$ for the two dimensional wedge diffraction problem is obtained by using (3) and (10) with the following result:

$$\begin{aligned} G_\rho(\rho'', \rho', k) &= i\int_0^\infty dT K_\rho(\rho'', \rho', T)e^{ik^2T} \\ &= \frac{1}{\alpha}\sum_{l=1}^{\infty}\sin\left(\frac{l\pi\phi''}{\alpha}\right)\sin\left(\frac{l\pi\phi'}{\alpha}\right) \\ &\quad \cdot\int_0^\infty \frac{dT}{T}\exp\left\{i\left(k^2T + \frac{\rho''^2+\rho'^2}{4T}\right)\right\} I_{l\pi/\alpha}\left(\frac{\rho''\rho'}{2Ti}\right). \end{aligned} \tag{11}$$

By using the following formula.

$$H_\nu^{(1)}(Z)J_\nu(z) = \frac{1}{\pi i}\int_0^{\infty i}\frac{dt}{t}\exp\left(\frac{t}{2} - \frac{Z^2+z^2}{2t}\right)\cdot I_\nu\left(\frac{Zz}{t}\right), \quad (|Z| \geq |z|), \tag{12}$$

well known modal summation formula[5] is obtained as follow:

$$G_\rho(\rho'', \rho', k) = \frac{\pi i}{\alpha}\sum_{l=1}^{\infty} H_{l\pi/\alpha}^{(1)}(k\rho_>)J_{l\pi/\alpha}(k\rho_<)\sin\left(\frac{l\pi\phi''}{\alpha}\right)\sin\left(\frac{l\pi\phi'}{\alpha}\right), \tag{13}$$

where $\rho_>$ and $\rho_<$ correspond the larger and smaller values of $\rho' = |\rho'|$ and $\rho'' = |\rho''|$, respectively.

5 Three Dimensional Wedge Diffraction

Three dimensional Feynman propagator $K(q'', q', T)$ is obtained by using (6), (7) and (10) with the result:

$$\begin{aligned} K(q'', q', T) &= \left(\frac{1}{4\pi Ti}\right)^{1/2}\frac{1}{\alpha Ti}\exp\left[\frac{i}{4T}\left\{\rho''^2+\rho'^2+(z''-z')^2\right\}\right] \\ &\quad \cdot\sum_{l=1}^{\infty}\sin\left(\frac{l\pi\phi''}{\alpha}\right)\sin\left(\frac{l\pi\phi'}{\alpha}\right) I_{l\pi/\alpha}\left(\frac{\rho''\rho'}{2Ti}\right). \end{aligned} \tag{14}$$

Substitution of (14) into (3) gives the solution of the problem;

$$\begin{aligned} G(q'', q', k) &= \frac{e^{-i\pi/4}}{\sqrt{4\pi}\alpha}\sum_{m=1}^{\infty} J_m \sin\left(\frac{m\pi\phi''}{\alpha}\right)\sin\left(\frac{m\pi\phi'}{\alpha}\right) \\ J_m &= \int_0^\infty dx x^{-3/2}\exp\left[i\left\{k^2x + \frac{\rho''^2+\rho'^2+(z''-z')^2}{4x}\right\}\right] I_{m\pi/\alpha}\left(\frac{\rho''\rho'}{2xi}\right). \end{aligned} \tag{15}$$

More familiar expression is obtained by employing the following identities of the Laplace transformation and the convolution theorem:

$$\int_0^\infty dT T^{-1/2}\exp\left\{iTy + \frac{i}{4T}(z''-z')^2\right\} = i\sqrt{\pi}e^{-i\pi/4}\frac{e^{i\sqrt{y}(z''-z')}}{\sqrt{y}} \tag{16}$$

$$\begin{aligned} &\int_0^\infty dT T^{-1}\exp\left\{iT(k^2-y) + \frac{i}{4T}(\rho''^2+\rho'^2)\right\} I_{l\pi/\alpha}\left(\frac{\rho''\rho'}{2Ti}\right) \\ &\qquad = \pi i H_{l\pi/\alpha}^{(1)}(\sqrt{k^2-y}\rho_>)J_{l\pi/\alpha}(\sqrt{k^2-y}\rho_<). \end{aligned} \tag{17}$$

The resultant familiar expression[5] of $G(q'', q', k)$ is as follow:

$$G(q'', q', k) = \frac{i}{2\alpha} \int_{-\infty}^{\infty} d\zeta e^{i\zeta(z''-z')}$$
$$\cdot \sum_{m=1}^{\infty} J_{m\pi/\alpha}(\sqrt{k^2-\zeta^2}\rho_<) H_{m\pi/\alpha}^{(1)}(\sqrt{k^2-\zeta^2}\rho_>) \sin\left(\frac{m\pi\phi''}{\alpha}\right) \sin\left(\frac{m\pi\phi'}{\alpha}\right). \quad (18)$$

6 Conclusions

Path integrals can be applied to the solution for wave problems of the Helmholz equation. As a demonstration, exact path integral solutions are constructed for the wedge diffraction problems of the Helmholtz equation.

References

[1] R.P.Feynman and A.R.Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, New York, 1965.

[2] L.S.Schulman, Techniques and Applications of Path Integration , Wiley-Interscience Publ., New York, 1981.

[3] C.Grosche, "δ-function perturbations and boundary problems by path integration," Ann. Physik, Vol. 2, pp.557-589, 1993.

[4] D.Peak and A.Inomata, "Summation over Feynman histories in polar coordinates," J.Math.Phys., Vol.10, p.1422, 1969.

[5] L.B.Felsen and N.Marcuvitz, Radiation and Scattering of Waves. IEEE Press, New York, 1994.

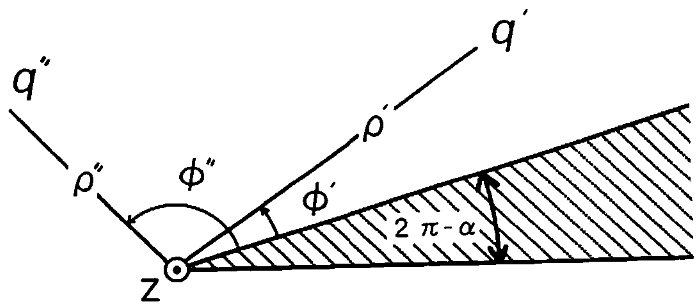

Fig.1 Wedge geometry