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Abstract—We discuss here three different systems, which address concepts of stochasticity and non-locality; human stick balancing, chase and escapes, and simple quantum mechanical problem. Though these problems are rather distant from each other, they show rich behaviors which are brought in by these two factors, separately or together.

1. Introduction

Fluctuations or noise in various systems have been of interest in various fields. In mathematics, for example, theory of probability has a long history. Effects of fluctuation or stochasticity on dynamical systems are also studied using stochastic differential equations, such as the Langevin equation. The concept of non-locality is less familiar. It has been the source of many "strange" phenomena of quantum mechanics. We take this concept a little wider in the sense that it also refers to interactions between two points separated by space and time. One such example is the self-feedback delay in controlling. Electric feedback circuits, physiological feedbacks are representatives.

Against this background, we consider these two concepts of stochasticity and non-locality through three examples in this short paper. In particular, we consider 1) human stick balancing, 2) a problem of chase and escape, and 3) a non-local equation for energy levels of a simple quantum square well potential. We try to convey by these simple examples that these factors of stochasticity and non-locality, separately or together, can lead to rather unexpected phenomena[1, 2, 3, 4].

2. Human stick balancing with fluctuations

Human stick balancing requires many factors. Recent experiments show that much of the corrective motion of the stick on the fingertips is faster than the human physiological feedback delay[5, 6]. This shows that there are more processes involved in this task than the feedback controls. Recently, an interesting observation was made. When a person rhythmically moved an object in one hand, balancing a stick in his or her other hand improved (Fig. 1(I)) [7, 8]. This was observed particularly with people who had intermediate balancing skills. We measured the time that they could keep the sticks balanced, and compared it with normal non-movement situations (Fig. 1(II)).

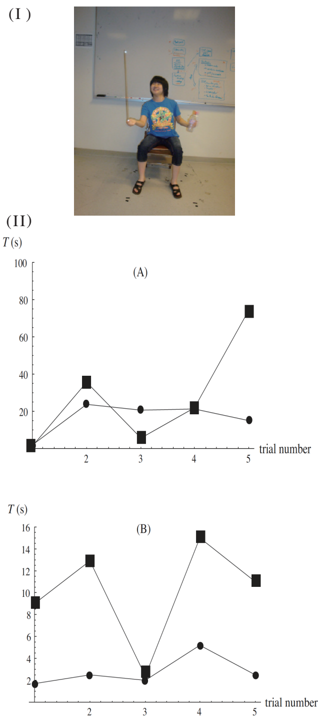

Figure 1: (I) Picture of subject balancing stick in one hand while moving object in other. (II) A comparison of the balancing time of the stick for five trials by two subjects (A) less practiced and (B) more practiced. The squares are with motions and the circles are without.

Following this line, other experiments were performed in which a person rhythmically moved his or her leg instead. These produced similar results. One hypothesis we formed from this was that an appropriate level of added fluctuating or rhythmic motion improved the balancing control with delayed feedbacks [9].

We posed another question concerning the nature of this fluctuation in improving the balancing control. Is it limited to physical noise? To address this question, other experiments were performed in which a person was asked to just imagine moving his or her leg during the stick balancing task. The results showed similar effects[10]. This implies that fluctuations in the level of intentions or thoughts may affect effectively during the stick balancing. Another hypothesis is that these fluctuations appropriately disrupt the feedback control loop. Relying too much on feedback control with human delay times could lead to less control during stick balancing tasks, and an appropriate level of intention diversion improves the control.

Even though we need to perform more experiments under a variety of conditions, from these results we believe that human control intricately involves various factors[11]. Also, these sets of experiment has shown the interplay between non-locality (in the form of feedback delay) and stochasticity.

3. Chases and Escapes

Mathematical interest in chase-escape problems has a long history and many interesting results have been obtained (for reviews see [12]). One famous problem is due to Hathaway of a dog (chaser) chasing a duck (escapee). In this problem the duck swims on a circular path with a constant velocity and the problem is to determine the dog's best strategy for pursuing and perhaps catching the duck. Typically, the dog points his velocity vector to the current position of the duck. However, in reality there is a delay in the dog's motion. Our preliminary results demonstrate that the inclusion of a state-dependent delay can make the dynamics of this pursuit-escape task very complex[13, 14].

Let us describe how we introduce a state-dependent time delay into the Hathaway's Problem. We assume that the tangent line of the dog's pursuit curve points not to the duck's present position as in the original model, but to its past position by a time delay τ. If τ is constant, then the problem can be mapped to a difference of the duck's initial position in the original problem. Thus the qualitative properties of the pursuit task are unchanged: a constant delay is equivalent to the introduction of a fixed phase shift. Here we consider more complex case in which τ is an increasing function proportional to the distance ρ between the dog and the duck. In concrete, we consider the simplest case of $\tau = \tau_0\rho$, where τ_0 is a scale factor. This reflects the situation that the dog's precision of detecting the escaping duck decreases as the distance between them increases.

Some results are shown in Figure 2, where we have varied τ_0 and the ratio n of the speed of the dog to the duck. We have found a variety of trajectories for the dog chasing the duck as follows.

(a) When $n < 1$, the dog cannot catch the duck, irrespective of the presence of delay (Fig. 2b without delay and Fig.2e with delay).

(b) When $n > 1$ with no delay, the dog can catch the duck (Fig. 2a). However, with the delay increased beyond a critical value, the dog cannot catch the duck (Figs. 2c, 2d, 2f). Also the trajectories of the dog can be quite complex (Figs. 2c-f).

Though these behaviors due to delay (temporal non-locality) needs to be analyzed further, the variety of dynamics are quite rich.

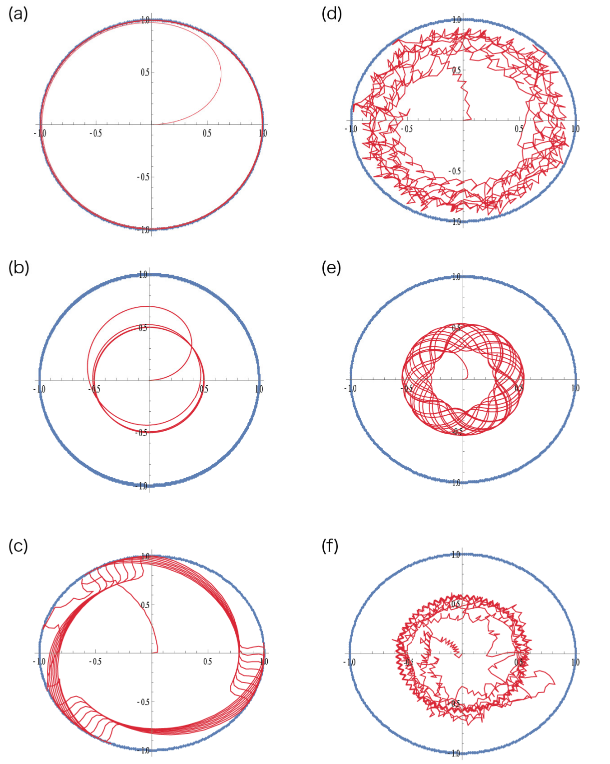

Figure 2: Examples of a circular chase (red) and escape (blue) with the distance dependent delays. The scale factor and speed ratio $[\tau_0, n]$ are given as $(a)[0, 1.01]$, $(b)[0, 0.5]$, $(c)[500, 1.01]$, $(d)[500, 1.5]$,$(e)[500, 0.5]$, $(f)[1050, 1.01]$, with the duck's velocity $v = 0.05$(The period to round the unit circle is $T \approx 126$.)

With respect to the stochasticity in the context of chases and escapes, we have considered fluctuations in the direction of the chasers' motions in the context of a "Group Chase and Escape"[15]. It has been found that certain level of such stochasticity can contribute to efficient catching events.

4. Non-local equation for quantum square-well energy levels

In quantum mechanics, stochasticity and non-locality are the two main elements which differs from classical mechanics[16, 17]. Here, we focus on non-locality, by which we mean to describe effects or dynamics that involve multiple points in space and/or time, and that cannot be composed simply by combining local effects. The quantum resonance tunneling is an example[18, 19]. The transmission rate of two potential barriers together, which are spatially separately placed, cannot be obtained correctly by simply "classically" combining the transmission rate of each single barrier. In fact, as is well known, the complete tunneling could be achieved theoretically by having two potential barriers, even though each barrier is not. Thus, we need to take into account effects of non-local two points separated in space all together. "Delayed Choice Experiments"[16, 20] is a example of temporal non-locality. We cannot describe outcomes of such experiments by a simple combination of what is measured or not measured at each local time point. Extended knowledge over time axes is needed, again, at once to describe the experimental results.

We take the following speculative view against this background by taking take non-locality at its "face value" and built it into dynamics of equation. Such a equation involves multiple points of space and time variables, which we term as a "non-local" equation in this paper as in [21, 22].

The equation we present in the following is the simplest first order non-local equation. In contrast to the Schrödinger equation, it contains only the first order derivative in space. The boundary points of the potential are explicitly included as non-local factors, which, in effect "replace" the boundary conditions. With a quantization rule of imposing oscillating dynamics inside the potential well, we show that it can reproduce quantized energy levels as given by the standard procedures of considering boundary conditions[23].

Let us start describing our equations. The first one is for the square well with infinite boundaries. The equation is simply given as follows.

$$\frac{d\mu(x)}{dx} = \begin{cases} (i)^{1+p}k\mu(x-\frac{L}{2}), & (0 \le x \le \frac{L}{2}) \\ (i)^{1+p}k\mu(x+\frac{L}{2}), & (-\frac{L}{2} \le x \le 0) \\ 0, & (\frac{L}{2} < |x|), \end{cases} \tag{1}$$

where $i = \sqrt{-1}$, $k = \frac{\sqrt{2mE}}{\hbar}$ with mass, m, and energy E of the quantum particle, and $\hbar = \frac{h}{2\pi}$, h being the Plank's constant. p is a parameter which takes values $0, 1$. The quantization condition is imposing a condition that, within the well, the function $\mu(x)$ admits only the oscillatory form. Namely,

$$\mu(x) \sim e^{i\omega x} \tag{2}$$

For $p = 0$, substituting this into the equation (1) yields,

$$i\omega = ik\cos(\frac{\omega L}{2}), \quad 0 = k\sin(\frac{\omega L}{2}). \tag{3}$$

These together leads to a quantization, $\omega^2 = k^2$ and $k_n = \frac{2n\pi}{L}, n = 1, 2, 3, \ldots$. The associated wave function can be constructed up to the normalization constant as

$$\psi(x) \sim \begin{cases} (\mu(x) - \mu^*(x))/2 & (-\frac{L}{2} \le x \le \frac{L}{2}) \\ 0 & (\frac{L}{2} < |x|) \end{cases} \tag{4}$$

Similarly, with $p = 1$, we obtain the other solution sets with $k_n = \frac{\pi}{L}(n+1), n = 0, 2, 4, \ldots$, with the associated wave function as

$$\psi(x) \sim \begin{cases} (\mu(x) + \mu^*(x))/2 & (-\frac{L}{2} \le x \le \frac{L}{2}) \\ 0 & (\frac{L}{2} < |x|) \end{cases} \tag{5}$$

These are well known results of the quantum bound states for this potential[24].

When the hight of potential is finite with $V_0 > E$ (Figure 3 (B)), we add a linear term in the Equation (1).

$$\frac{d\mu(x)}{dx} = \begin{cases} -\alpha\mu(x) & (\frac{L}{2} < x) \\ -\alpha\mu(x) + (i)^{1+p}\gamma\mu(x-\frac{L}{2}) & (0 \le x \le \frac{L}{2}) \\ +\alpha\mu(x) + (i)^{1+p}\gamma\mu(x+\frac{L}{2}) & (-\frac{L}{2} \le x \le 0) \\ +\alpha\mu(x) & (x < -\frac{L}{2}), \end{cases} \tag{6}$$

where $\alpha = \frac{\sqrt{2m(V_0-E)}}{\hbar}$ and $\gamma = \frac{\sqrt{2mV_0}}{\hbar}$. We note an ordinary relation between these parameters, $k^2 + \alpha^2 = \gamma^2$.

By going through the same procedure of imposing the condition of Equation (2), we obtain the sets of equations for $p = 0, 1$.

For $p = 0$,

$$i\omega = i\gamma\cos(\frac{\omega L}{2}), \quad \alpha = \gamma\sin(\frac{\omega L}{2}), \tag{7}$$

leading to $\omega^2 + \alpha^2 = \gamma^2$ and

$$\frac{\alpha L}{2} = \frac{\omega L}{2}\tan(\frac{\omega L}{2}) \tag{8}$$

For $p = 1$,

$$i\omega = i\gamma\sin(\frac{\omega L}{2}), \quad \alpha = -\gamma\cos(\frac{\omega L}{2}), \tag{9}$$

leading again to $\omega^2 + \alpha^2 = \gamma^2$, and

$$\frac{\alpha L}{2} = -\frac{\omega L}{2}\cot(\frac{\omega L}{2}) \tag{10}$$

By identifying $k = \omega$, Eqs. (8,10) give the standard quantum energy levels for this potential.

Also, the associated wave functions can be constructed, for $p = 0, 1$,

$$\psi(x) \sim \begin{cases} e^{+\alpha x} & (x < -\frac{L}{2}) \\ (\mu(x) + (-1)^p\mu^*(x))/2 & (-\frac{L}{2} \le x \le \frac{L}{2}) \\ e^{-\alpha x} & (\frac{L}{2} < x) \end{cases} \tag{11}$$

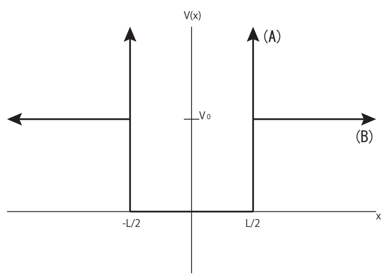

Figure 3: Quantum square well with infinite (A) and finite (B) barrier height.

Normally, quantizations with square well potentials are done through physical considerations at the boundaries. Here, in a sense, boundaries of the potential are incorporated into the equation itself as a non-local element, and the quantization condition is given by requirements of the oscillatory nature of the solution. It is yet to be investigated that this type of approach can be developed for obtaining or approximating quantum bound states for more general types of potentials.

5. Discussion

These examples we have shown above are quite different in many aspects. However, they all contain elements of stochasticity and/or non-locality. Though these are simple examples, some intricate behaviors, which are not yet clearly understood, have been observed. We hope that this short paper convey some of these intriguing aspects of stochasticity and non-locality, which are yet to be explored.

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[24] Note that the wave function $\psi(x)$ itself does not satisfy the differential equation.