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A New Continuous-Time Algorithm for Calculating Algebraic Connectivity of Multi-Agent Networks

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Abstract—How to maintain the connectivity is an important issue in mobile agent networks. In this paper, we propose a new continuous-time algorithm for calculating the algebraic connectivity of the graph representing the interaction between agents. This is simpler than the conventional algorithm in the sense that less multiplications are needed. We study the dynamical behavior of the proposed algorithm and prove that it can find the algebraic connectivity of the graph for almost all initial conditions.

1. Introduction

Mobile agent networks have found many applications such as target tracking, formation control and environmental monitoring [1]. Each agent in a network not only moves but also interacts with other agents to collect information obtained by sensors. It is thus an important issue for mobile agent networks how to maintain their connectivity.

A promising approach to maintaining the connectivity is to compute the second smallest eigenvalue of the Laplacian, or the algebraic connectivity [2], of the graph representing the interaction between agents. Since this quantity is positive if and only if the graph is connected, the connectivity is maintained if agents move so that its value is kept positive. Recently, Yang *et al.* [3] proposed a continuous-time algorithm for calculating the algebraic connectivity of the graph. Their algorithm has an advantage that the calculation can be performed in a decentralized manner.

In this paper, we propose a new continuous-time algorithm for calculating the algebraic connectivity of the graph representing the interaction between agents. The proposed algorithm is simpler than the one of Yang *et al.* in the sense that less multiplications are needed. The reduction of the number of multiplications is important for implementation. We study dynamical properties of the proposed algorithm theoretically. In particular, we prove that the proposed algorithm can find the algebraic connectivity of the graph for almost all initial conditions.

2. Algebraic Connectivity of Multi-Agent Networks

We consider a network of n agents in which each agent can interact only with a small number of other agents. The interaction between agents can be expressed by a simple undirected graph $G = (V, E)$ where $V = \{1, 2, \ldots, n\}$ is the set of vertices representing n agents and E is the set of edges which are represented as unordered pairs of distinct vertices. A pair $\{i, j\}$ is a member of E if and only if agents i and j can interact with each other.

The adjacency matrix $A = (a_{ij})$ and the degree matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ of G are defined by

$$a_{ij} = \begin{cases} 1, & \text{if } \{i, j\} \in E \\ 0, & \text{otherwise} \end{cases}$$

and

$$d_i = \sum_{j=1}^{n} a_{ij}, \quad i = 1, 2, \ldots, n$$

respectively. With these notations, the Laplacian matrix L of G can be defined by $L = D - A$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of L. Since L is real and symmetric, its eigenvalues are all real. In the following, we assume without loss of generality that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then the following statements hold true.

1. $\lambda_1 = 0$ and $\mathbf{1} = (1, 1, \ldots, 1)^{\mathrm{T}}$ is the eigenvector corresponding to λ_1.
2. $\lambda_2 > 0$ if and only if G is connected.

It follows from the second property that we can determine whether the graph G is connected or not by calculating the second smallest eigenvalue λ_2 of L. In particular, the value of λ_2 plays an important role for maintaining the connectivity of mobile agent networks. In algebraic graph theory, the second smallest eigenvalue of L is called the algebraic connectivity [2] of the graph G. So we hereafter use this terminology.

Throughout this paper, we assume for simplicity that the algebraic connectivity λ_2 is simple, that is, λ_2 is not a repeated eigenvalue of L. This implies that λ_2 is positive.

Since L is real and symmetric, there exists an orthonormal matrix P such that

$$PLP^{\mathrm{T}} = \mathrm{diag}(0, \lambda_2, \ldots, \lambda_n) \triangleq L^* .$$

The i-th column of P^{T} is denoted by $\boldsymbol{p}_i$. Then P^{T} is expressed as $P^{\mathrm{T}} = (\boldsymbol{p}_1 \boldsymbol{p}_2 \cdots \boldsymbol{p}_n)$. Furthermore, the j-th element of $\boldsymbol{p}_i$ is denoted by p_{ij}. This notation is very natural because the (i, j) element of P is denoted by p_{ij}. Since λ_2 is simple, $\boldsymbol{p}_2$ is an eigenvector of L corresponding to λ_2. Also, $\boldsymbol{p}_1$ is either $\frac{1}{\sqrt{n}}\mathbf{1}$ or $-\frac{1}{\sqrt{n}}\mathbf{1}$.

3. Previous Result

Yang *et al.* [3] proposed a continuous-time algorithm for calculating the algebraic connectivity of the graph representing the interaction between agents. In this algorithm, the i-th agent changes its state x_i according to

$$\dot{x}_i = -k_1\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right) - k_2\sum_{j=1}^{n} a_{ij}(x_i - x_j) - k_3\left(\frac{1}{n}\sum_{j=1}^{n} x_j^2 - 1\right)x_i \tag{1}$$

where k_1, k_2 and k_3 are positive constants. By introducing $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, we can rewrite (1) in a vector form as

$$\dot{\boldsymbol{x}} = -k_1\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)\mathbf{1} - k_2 L\boldsymbol{x} - k_3\left(\frac{1}{n}\sum_{j=1}^{n} x_j^2 - 1\right)\boldsymbol{x}. \tag{2}$$

In the above differential equations, it is assumed that the averages $\frac{1}{n}\sum_{j=1}^{n} x_j$ and $\frac{1}{n}\sum_{j=1}^{n} x_j^2$ can be obtained instantaneously, while each agent cannot interact with all of the other agents in general. However, this assumption can be satisfied approximately by using a consensus algorithm [1, 4] with a much smaller time constant than the main algorithm (2).

Yang *et al.* analyzed the dynamical behavior of (2) and derived the following theorem.

Theorem 1 ([3]) Suppose that the initial value $\boldsymbol{x}(0)$ satisfies $\boldsymbol{p}_2^{\mathrm{T}}\boldsymbol{x}(0) \neq 0$. Then any solution $\boldsymbol{x}(t)$ of (2) converges to either $\mu\boldsymbol{p}_2$ or $-\mu\boldsymbol{p}_2$ where μ is a positive constant given by

$$\mu = \sqrt{\frac{n(k_3 - k_2\lambda_2)}{k_3}}$$

if and only if positive constants k_1, k_2 and k_3 satisfy the following conditions.

$$k_1 > k_2\lambda_2 \tag{3}$$

$$k_3 > k_2\lambda_2 \tag{4}$$

From Theorem 1 we can easily see that

$$\lim_{t\to\infty}\frac{k_3}{k_2}\left(1 - \frac{1}{n}\sum_{i=1}^{n} x_i(t)^2\right) = \frac{k_3}{k_2}\left(1 - \frac{\mu^2}{n}\right) = \lambda_2$$

which means that the algebraic connectivity λ_2 can be obtained from the solution $\boldsymbol{x}(t)$ of (2).

4. New Algorithm and its Convergence

4.1. Proposed Algorithm

In this paper, we propose a new continuous-time algorithm described by

$$\dot{\boldsymbol{x}} = -k_1\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)\mathbf{1} - k_2 L\boldsymbol{x} - k_3\left(\frac{1}{n}\sum_{j=1}^{n} |x_j| - 1\right)\boldsymbol{x}. \tag{5}$$

This algorithm is identical with (2) except the third term of the right-hand side. To be more specific, x_j^2 in (2) is replaced with $|x_j|$ in (5). It is interesting from a theoretical point of view to see whether or not this replacement will affect the convergence of the algorithm. Also, this replacement is very important from a practical point of view. In fact, (5) can be implemented by a simpler circuit than (2) because the former requires less multipliers than the latter.

Introducing a new variable $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$ which is defined by $\boldsymbol{y} \triangleq P\boldsymbol{x}$, (5) can be rewritten as

$$\begin{aligned}\dot{\boldsymbol{y}} &= -k_1\mathrm{diag}(1, 0, \ldots, 0)\boldsymbol{y} - k_2 L^*\boldsymbol{y} - k_3\left(\frac{\|P^{\mathrm{T}}\boldsymbol{y}\|_1}{n} - 1\right)\boldsymbol{y} \\ &= -k_2\tilde{L}^*\boldsymbol{y} - k_3\left(\frac{\|P^{\mathrm{T}}\boldsymbol{y}\|_1}{n} - 1\right)\boldsymbol{y}\end{aligned} \tag{6}$$

where $\tilde{L}^* = \mathrm{diag}(k_1/k_2, \lambda_2, \lambda_3, \ldots, \lambda_n)$. In the following, the i-th diagonal element of $\tilde{L}^*$ is denoted by $\tilde{\lambda}_i$ for the sake of notational simplicity, that is, $\tilde{\lambda}_1 = k_1/k_2$ and $\tilde{\lambda}_i = \lambda_i$ for $i = 2, 3, \ldots, n$.

4.2. Equilibrium Points and Their Stability

The goal of this subsection is to specify all equilibrium points of (6) and their local stability.

Lemma 1 Suppose that diagonal elements of $\tilde{L}^*$ are all distinct. Then the set of all equilibrium points of (6) is given by $\{\mathbf{0}\} \cup \{\pm\boldsymbol{y}_i \,|\, i \in I\}$ where $I \triangleq \{i \,|\, k_3 > k_2\tilde{\lambda}_i\} \subseteq \{1, 2, \ldots, n\}$ and $\boldsymbol{y}_i$ ($i \in \{1, 2, \ldots, n\}$) is the vector such that the j-th element is given by

$$(\boldsymbol{y}_i)_j = \begin{cases} n(k_3 - k_2\tilde{\lambda}_i)/(k_3\|\boldsymbol{p}_i\|_1), & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

In particular, $\boldsymbol{y}_2$ and $-\boldsymbol{y}_2$ are equilibrium points of (6) if and only if (4) is satisfied.

Proof: It is apparent from (6) that $\mathbf{0}$ is an equilibrium point. Setting the right-hand side of (6) to zero, we have

$$\tilde{L}^*\boldsymbol{y} = -\frac{k_3}{k_2}\left(\frac{\|P^{\mathrm{T}}\boldsymbol{y}\|_1}{n} - 1\right)\boldsymbol{y} \tag{7}$$

which means that other equilibrium points are restricted to eigenvectors of $\tilde{L}^*$. Let $\boldsymbol{y}_i$ be an eigenvector corresponding to the i-th eigenvalue $\tilde{\lambda}_i$ of $\tilde{L}^*$. Then the i-th element $(\boldsymbol{y}_i)_i$ takes a nonzero value, say α_i, and other elements vanish. In the following, we assume without loss of generality that α_i is positive. Substituting $\boldsymbol{y} = \boldsymbol{y}_i = (0, \ldots, 0, \alpha_i, 0, \ldots, 0)^{\mathrm{T}}$ into (7), we have

$$|\alpha_i| = \alpha_i = \frac{n(k_3 - k_2\tilde{\lambda}_i)}{k_3\|\boldsymbol{p}_i\|_1}. \tag{8}$$

Therefore, the eigenvector $\boldsymbol{y}_i$ satisfies (7) if and only if the right-hand side is positive, that is, $k_3 > k_2\tilde{\lambda}_i$. This completes the proof for the first part. The second part is obvious from the first part. □

Theorem 2 Suppose that diagonal elements of $\tilde{L}^*$ are all distinct. If positive constants k_1, k_2 and k_3 satisfy (3) and (4) then $\boldsymbol{y}_2$ and $-\boldsymbol{y}_2$ are asymptotically stable equilibrium points of (6) and all other equilibrium points are unstable.

Proof: We first show that $\boldsymbol{0}$ is unstable. Let us consider the solution $\boldsymbol{y}(t)$ of (6) starting from $(0, \epsilon_2, 0, \ldots, 0)^{\mathrm{T}}$ where ϵ_2 is a positive constant. Since $y_i(t) = 0$ holds for all $t \geq 0$ and all i except 2, it suffices for us to analyze the dynamical behavior of $y_2(t)$ which obeys the differential equation:

$$\dot{y}_2 = \left(k_3 - \lambda_2 k_2 - \frac{k_3\|\boldsymbol{p}_2\|_1}{n}|y_2|\right) y_2 . \qquad (9)$$

Note that $k_3 - \lambda_2 k_2$ is positive from the assumption (4). Hence the quantity in the parenthesis of the right-hand side is positive if and only if $|y_2| < n(k_3 - \lambda_2 k_2)/(k_3\|\boldsymbol{p}_2\|_1)$. This means that $y_2(t)$ moves away from 0 if its initial value ϵ_2 is sufficiently small. Therefore, the origin $\boldsymbol{0}$ is unstable.

We next assume that $k_3 - k_2\tilde{\lambda}_1 = k_3 - k_1 > 0$ in addition to (3) and (4) and show that $\boldsymbol{y}_1$ is an unstable equilibrium points of (6). Let us consider the solution $\boldsymbol{y}(t)$ of (6) starting from $(\alpha_1 + \epsilon_1, \epsilon_2, 0, \ldots, 0)^{\mathrm{T}}$ where ϵ_1 and ϵ_2 are positive constants. Since $y_i(t) = 0$ holds for all $t \geq 0$ and all i except 1 and 2, it suffices for us to analyze the dynamical behavior of $y_1(t)$ and $y_2(t)$ which obey the set of differential equation:

$$\begin{aligned}
\dot{y}_1 &= -k_1 y_1 - k_3\left(\frac{\|y_1\boldsymbol{p}_1 + y_2\boldsymbol{p}_2\|_1}{n} - 1\right) y_1 \\
\dot{y}_2 &= -k_2\lambda_2 y_2 - k_3\left(\frac{\|y_1\boldsymbol{p}_1 + y_2\boldsymbol{p}_2\|_1}{n} - 1\right) y_2
\end{aligned}$$

Here, we should note that

$$\begin{aligned}
&-k_2\lambda_2 - k_3\left(\frac{\|y_1\boldsymbol{p}_1 + y_2\boldsymbol{p}_2\|_1}{n} - 1\right) \\
&\geq (k_1 - k_2\lambda_2) - \frac{k_3}{n}(\|(y_1 - \alpha_1)\boldsymbol{p}_1\|_1 + \|y_2\boldsymbol{p}_2\|_1) .
\end{aligned}$$

Since $k_1 - k_2\lambda_2$ is positive from the assumption (3), the above quantity is positive if and only if $|y_1 - \alpha_1| \times \|\boldsymbol{p}_1\|_1 + |y_2| \times \|\boldsymbol{p}_2\|_1 < n(k_1 - k_2\lambda_2)/k_3$. This means that $y_2(t)$ moves away from 0 if ϵ_1 and ϵ_2 are sufficiently small. Therefore, the equilibrium point $\boldsymbol{y}_1$ is unstable.

In the same way as above, we can show that the equilibrium points $\boldsymbol{y}_i$ and $-\boldsymbol{y}_i$ $(i = 3, 4, \ldots, n)$ are, if they exist, unstable.

We finally show that $\boldsymbol{y}_2$ is stable. It is obvious that the right-hand side of (6) is locally Lipschitz continuous. Let

$$V(\boldsymbol{y}) \triangleq \frac{1}{2}(\boldsymbol{y} - \boldsymbol{y}_2)^{\mathrm{T}} C (\boldsymbol{y} - \boldsymbol{y}_2)$$

where $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$ is a positive diagonal matrix of which the diagonal elements are given by

$$\begin{aligned}
c_1 &= \frac{n^2}{k_1 - k_2\lambda_2} && (10) \\
c_2 &= \frac{1}{n(k_3 - k_2\lambda_2)} && (11) \\
c_i &= \frac{n^2}{k_2(\lambda_i - \lambda_2)} \quad (i = 3, 4, \ldots, n) && (12)
\end{aligned}$$

Then we immediately see that $V(\boldsymbol{y}_2) = 0$ and $V(\boldsymbol{y}) > 0$ for all $\boldsymbol{y} \neq \boldsymbol{y}_2$. We will show in the following that there exists a domain $\mathcal{D}$ containing $\boldsymbol{y}_2$ such that $\dot{V}(\boldsymbol{y}) < 0$ for all $\boldsymbol{y} \in \mathcal{D} - \{\boldsymbol{y}_2\}$. If this is true, we can conclude from [5, Theorem 3.1] that $\boldsymbol{y}_2$ is asymptotically stable.

Let us define two index sets J_1 and J_2 by

$$\begin{aligned}
J_1 &= \{ j \mid p_{2j} = 0 \} \\
J_2 &= \{1, 2, \ldots, n\} - J_1
\end{aligned}$$

If $\boldsymbol{y}$ is sufficiently close to $\boldsymbol{y}_2$, that is, y_2 is sufficiently close to α_2 and $|y_i|$ is sufficiently small for all i except 2, then $\|P^{\mathrm{T}}\boldsymbol{y}\|_1$ can be rewritten as follows:

$$\|P^{\mathrm{T}}\boldsymbol{y}\|_1 = y_2\|\boldsymbol{p}_2\|_1 + b$$

where

$$\begin{aligned}
b &= b(y_1, y_3, \ldots, y_n) \\
&= \sum_{j \in J_1}\left| y_1 p_{1j} + \sum_{i=3}^{n} y_i p_{ij} \right| \\
&\quad + \sum_{j \in J_2} \mathrm{sgn}(p_{2j})\left(y_1 p_{1j} + \sum_{i=3}^{n} y_i p_{ij} \right) . \qquad (13)
\end{aligned}$$

The time derivative of $V(\boldsymbol{y})$ along the solution of (6) around $\boldsymbol{y}_2$ can be calculated as follows:

$$\begin{aligned}
\dot{V}(\boldsymbol{y}) &= (\boldsymbol{y} - \boldsymbol{y}_2)^{\mathrm{T}} C \left\{ -k_2\tilde{L}^*\boldsymbol{y} - k_3\left(\frac{\|P^T\boldsymbol{y}\|_1}{n} - 1\right)\boldsymbol{y} \right\} \\
&= (\boldsymbol{y} - \boldsymbol{y}_2)^{\mathrm{T}} C \left\{ -k_2\tilde{L}^*\boldsymbol{y} - k_3\left(\frac{y_2\|\boldsymbol{p}_2\|_1 + b}{n} - 1\right)\boldsymbol{y} \right\} \\
&= -c_1\left(k_1 - k_3 + \frac{k_3\|\boldsymbol{p}_2\|_1\alpha_2}{n}\right) y_1^2 \\
&\quad - c_2\left(k_2\lambda_2 - k_3 + \frac{2k_3\|\boldsymbol{p}_2\|_1\alpha_2}{n}\right)(y_2 - \alpha_2)^2 \\
&\quad - c_2\left(k_2\lambda_2 - k_3 + \frac{k_3 b}{n} + \frac{k_3\|\boldsymbol{p}_2\|_1\alpha_2}{n}\right)\alpha_2(y_2 - \alpha_2) \\
&\quad - \sum_{i=3}^{n} c_i\left(k_2\lambda_i - k_3 + \frac{k_3\|\boldsymbol{p}_2\|_1\alpha_2}{n}\right) y_i^2 \\
&\quad + O(\|\boldsymbol{y} - \boldsymbol{y}_2\|_1^3) \qquad (14)
\end{aligned}$$

Since $k_3\|\boldsymbol{p}_2\|_1\alpha_2/n = k_3 - k_2\lambda_2$ follows from (8), (14) can be rewritten as

$$\begin{aligned}
\dot{V}(\boldsymbol{y}) &= -c_1(k_1 - k_2\lambda_2)y_1^2 - c_2(k_3 - k_2\lambda_2)(y_2 - \alpha_2)^2 \\
&\quad - c_2\frac{k_3 b}{n}\alpha_2(y_2 - \alpha_2) - \sum_{i=3}^{n} c_i k_2(\lambda_i - \lambda_2) y_i^2 \\
&\quad + O(\|\boldsymbol{y} - \boldsymbol{y}_2\|_1^3)
\end{aligned}$$

Since $|b| \leq 2n(|y_1| + \sum_{i=3}^{n} |y_i|)$ follows from (13) and $\alpha_2 \leq n(k_3 - k_2\lambda_2)/k_3$ follows from (8), we further have

$$\begin{aligned} \dot{V}(\boldsymbol{y}) \quad \leq \quad & -c_1(k_1 - k_2\lambda_2)y_1^2 - c_2(k_3 - k_2\lambda_2)(y_2 - \alpha_2)^2 \\ & + 2c_2 n(k_3 - k_2\lambda_2)|y_2 - \alpha_2| \left(|y_1| + \sum_{i=3}^{n} |y_i| \right) \\ & - \sum_{i=3}^{n} c_i k_2(\lambda_i - \lambda_2) y_i^2 + O(\|\boldsymbol{y} - \boldsymbol{y}_2\|_1^3) \qquad (15) \end{aligned}$$

Substituting (10)–(12) into the right-hand side, we have

$$\begin{aligned} \dot{V}(\boldsymbol{y}) \quad \leq \quad & -n^2|y_1|^2 - \frac{1}{n}|y_2 - \alpha_2|^2 \\ & + 2|y_2 - \alpha_2| \left(|y_1| + \sum_{i=3}^{n} |y_i| \right) - n^2 \sum_{i=3}^{n} |y_i|^2 \\ & + O(\|\boldsymbol{y} - \boldsymbol{y}_2\|_1^3) \\ = \quad & -\boldsymbol{\delta}^{\mathrm{T}} F \boldsymbol{\delta} + O(\|\boldsymbol{\delta}\|_1^3) \end{aligned}$$

where $\boldsymbol{\delta} = (|y_1|, |y_2 - \alpha_2|, |y_3|, \ldots, |y_n|)^{\mathrm{T}}$ and

$$F = \begin{pmatrix} n^2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 1/n & -1 & -1 & \cdots & -1 \\ 0 & -1 & n^2 & 0 & \cdots & 0 \\ 0 & -1 & 0 & n^2 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & -1 & 0 & \cdots & 0 & n^2 \end{pmatrix}.$$

Here we can easily show that F is positive definite by using Sylvester's criterion. Therefore, $\dot{V}(\boldsymbol{y})$ is negative if $\boldsymbol{y}$ is sufficiently close to $\boldsymbol{y}_2$. □

4.3. Global Convergence

We will show in this subsection that for almost all initial conditions the solution of (6) converges to either $\boldsymbol{y}_2$ or $-\boldsymbol{y}_2$. This implies that the solution of (5) converges to either $\alpha_2 \boldsymbol{p}_2 = \{n(k_3 - k_2\lambda_2)/(k_3\|\boldsymbol{p}_2\|_1)\}\boldsymbol{p}_2$ or $-\alpha_2\boldsymbol{p}_2$. As in the case of (2), the algebraic connectivity λ_2 can be obtained from the solution $\boldsymbol{x}(t)$ of (5) because the following holds.

$$\lim_{t\to\infty} \frac{k_3}{k_2}\left(1 - \frac{\|\boldsymbol{x}(t)\|_1}{n}\right) = \frac{k_3}{k_2}\left(1 - \frac{\alpha_2\|\boldsymbol{p}\|_1}{n}\right) = \lambda_2$$

Lemma 2 Let $\boldsymbol{y}(t) = (y_1(t), y_2(t), \ldots, y_n(t))^{\mathrm{T}}$ be any solution of (6). Then $|y_i(t)| \leq \max\{|y_i(0)|, n\}$ holds for all $t \geq 0$ and all $i \in \{1, 2, \ldots, n\}$.

Proof: If $|y_i(t)| \geq n$ then we have $\|P^{\mathrm{T}}\boldsymbol{y}(t)\|_1 \geq \|P^{\mathrm{T}}\boldsymbol{y}(t)\|_2 = \|\boldsymbol{y}(t)\|_2 \geq n$. Substituting this inequality into (6), we have

$$\dot{y}_i(t) \begin{cases} \leq -k_2\tilde{\lambda}_i y_i(t) \leq -k_2\tilde{\lambda}_i n < 0, & \text{if } y_i(t) \geq n \\ \geq -k_2\tilde{\lambda}_i y_i(t) \geq k_2\tilde{\lambda}_i n > 0, & \text{if } y_i(t) \leq -n \end{cases}$$

Therefore, if $|y_i(0)| \geq n$ then $|y_i(t)|$ decreases monotonically as long as $|y_i(t)| \geq n$ and reaches n at some time $t_0 \in [0, \infty)$. Furthermore, $|y_i(t)| < n$ holds for all $t \geq t_0$. □

The convergence property of the proposed algorithm (5) is stated as follows.

Theorem 3 Suppose that diagonal elements of $\tilde{L}^*$ are all distinct. Suppose also that the initial value $\boldsymbol{y}(0)$ is not an equilibrium point of (6) and $y_2(0) \neq 0$. Then any solution $\boldsymbol{y}(t)$ of (6) converges to either $\boldsymbol{y}_2$ or $-\boldsymbol{y}_2$ if positive constants k_1, k_2 and k_3 satisfy the conditions (3) and (4).

Proof: Let i be any member of $\{1, 2, \ldots, n\} - \{2\}$. If $y_i(0) = 0$ then $y_i(t) = 0$ for all $t \geq 0$. If $y_i(0) \neq 0$ then $y_2(t)/y_i(t)$ goes to infinity because

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln \left| \frac{y_2}{y_i} \right| = \frac{\dot{y}_2}{y_2} - \frac{\dot{y}_i}{y_i} = k_2(\tilde{\lambda}_i - \lambda_2) > 0\,.$$

On the other hand, by Lemma 2, $y_2(t)$ is bounded for all $t \geq 0$. These two facts imply that $y_i(t)$ converges to zero. Therefore, for sufficiently large t, $y_2(t)$ obeys the differential equation (9). It is easily seen that the solution $y_2(t)$ of (9) converges either α_2 or $-\alpha_2$. Therefore, the solution $\boldsymbol{y}(t)$ of (6) converges to either $\boldsymbol{y}_2$ or $-\boldsymbol{y}_2$. □

5. Conclusion

We have proposed a new continuous-time algorithm for calculating the algebraic connectivity of mobile agent networks. The proposed algorithm is simpler than the one of Yang *et al.* but has the same convergence property. In future works, we will study the dynamics of the proposed algorithm for the case where the averages $\frac{1}{n}\sum_{i=1}^{n} x_i$ and $\frac{1}{n}\sum_{i=1}^{n} |x_i|$ are not obtained in real-time but estimated with a continuous-time consensus algorithm.

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