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Visualization analysis on stretch-and-fold mechanism of chaotic attractors

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Abstract—We propose a simple visualization method for detecting the stretch-and-fold mechanism in chaotic dynamical systems. In the proposed method, we first place a rectangle that is centered at a point on the trajectory of a chaotic attractor and then uniformly arrange points on the rectangle. Next, applying the dynamics and observing the temporal evolution of the center point and the uniformly arranged points on the rectangle, we track the temporal evolution of the distances between the center point and the points on the rectangle. Finally, we express them using multiple colors and draw the colors representing the distances at the initial position on the rectangle. Then, tracking the temporal changes of these colors generated by the temporal evolution of the distances, we can observe the appearance of stripe patterns on the rectangle over time in the case of chaotic dynamics. We show that our method can detect the stretch-and-fold mechanism in chaotic dynamics by the stripe pattern. In addition, the stripe patterns are evaluated quantitatively.

1. Introduction

Several complex phenomena that exist in the real world might be produced from a deterministic nonlinear, possibly chaotic, dynamical system. In this sense, it is one of fundamental issues to elucidate mathematical structures of chaotic dynamics toward understanding of the complex phenomena and various technological applications.

The orbital instability and nonlinear folding are one of the basic and important features of chaotic dynamics. Even though two nearby trajectories separate exponentially in chaotic dynamical systems, they either remain within a finite phase space. These features are realized by the stretch-and-fold mechanism in chaotic dynamics [1]. Without the stretch-and-fold mechanism, chaotic attractors cannot be observed because two nearby trajectories simply diverge.

To analyze the stretch-and-fold mechanism, several methods have been proposed. The methods based on the Poincaré section are the most basic tool to visualize the stretch-and-fold mechanism in chaotic dynamics (see for example, [2, 3, 4, 5, 6, a]). If we appropriately arrange the Poincaré section, we can effectively visualize the stretch-and-fold mechanism. However, it is not so easy to arrange the sections on the chaotic attractors because of the variety of spatial structures of the chaotic attractors even in a low-dimensional state space.

Another possible approach is visualization of the long time evolution of points on the attractor. In this approach, a small sphere is prepared in a state space, and points are arranged on the surface of the small sphere. Then, the points on the small sphere are evolved by the rules of a dynamical system. The temporal evolution of the small sphere is tracked over time. However, if the temporal evolution of the small sphere is directly visualized, the points on the small sphere gradually distribute on the chaotic attractor, then the stretch-and-fold mechanism cannot be visualized because of the long term unpredictability of chaotic dynamics. To improve this point, a simple effective visualization method which is different from the conventional ones have been proposed in Refs. [7, b]. In the method, a small hypersphere which is centered at a point on a trajectory in the state space of the chaotic dynamics is prepared, and points are uniformly arranged on the surface of the small hypersphere. Then, the points on the hypersphere are evolved by the evolution rule of the chaotic dynamics and tracked over time. The key point of the method in Ref. [7] is that we observe a temporal change in the distances between the center point and the points on the surface of the hypersphere not on the attractor but on the initial position of the hypersphere. The temporal changes in the distances are described by multiple colors, which the points on the initial hypersphere are colored based on the distances. If the dynamics is chaotic, the stretch-and-fold mechanism is visualized by a stripe pattern on the initial hypersphere.

In this paper, we investigate quantitatively how the stripe pattern changes over time and thereby distinguish the dynamic regimes of the dynamical systems. In Ref. [7], to visualize the stretch-and-fold mechanism, the hypersphere is used. To investigate the temporal changes of the stripe pattern, we here use a more simple object — rectangle — instead of the hypersphere. This simple improvement enables us to evaluate the stripe patterns qualitatively. In numerical experiments, we apply the proposed method to three-dimensional chaotic dynamical systems and show that the stripe patterns can be observed even on the rectangle, or a two-dimensional image. In addition, we show that the stripe patterns formed by the chaotic dynamics have some characteristic properties by the numerical simulations.

2. Method

Let a point on an n-dimensional phase space at time t be $\boldsymbol{x}(t) = (x_1(t),\ x_2(t),\ \ldots,\ x_n(t)),\ (t = 1, \ldots, T)$. In our method, we firstly place a rectangle along the trajectory of a chaotic dynamical system so that its center is a point $\boldsymbol{x}(t_0)$ on the trajectory. Next, m points are arranged on the rectangle and they are evolved by the dynamics. Tracking temporal changes in the distances between the center point and the points on the rectangle, we can visualize the stretch-and-fold mechanism. The algorithm is shown in the following:

1. At the initial time t_0, a rectangle which is centered at $\boldsymbol{x}(t_0)$ $(\in \mathbb{R}^n)$ is prepared. The rectangle is a square l on a side, and the parameter l is set to 5% of the maximum distance between points on an attractor through this paper. The point $\boldsymbol{x}(t_0)$ is included in the trajectory of the n-dimensional dynamical system defined by $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$. Then, m points $\boldsymbol{r}_1(t_0), \boldsymbol{r}_2(t_0), \ldots \boldsymbol{r}_m(t_0)$ are uniformly arranged on the rectangle.

2. The points $\boldsymbol{r}_i(t)$ $(i = 1, \ldots, m)$ are evolved by the dynamical system. The time t is increased by Δt, such that $t \leftarrow t + \Delta t$. We defined the ith time t_i as $t_i \equiv t_0 + i\Delta t$.

3. The distances between the center point $\boldsymbol{x}(t)$ and the points $\boldsymbol{r}_i(t)$ are calculated by $d_i(t) = \|\boldsymbol{x}(t) - \boldsymbol{r}_i(t)\|$, where $\|\cdot\|$ shows the Euclidean norm. The point $\boldsymbol{r}_i(t)$ is then colored by the distance $d_i(t)$. In addition, the points $\boldsymbol{r}_i(t_0)$ $(i = 1, \ldots, m)$ which maintains the initial rectangle shape are also colored by $d_i(t)$.

4. Steps 2 and 3 are repeated.

The points on trajectories are usually distributed over the chaotic attractor as time evolves. However, in the step 3, coloring the points $\boldsymbol{r}_i(t_0)$ on the initial rectangle shape by the distance $d_i(t)$, we can observe how the rectangle is evolved with time through the temporal changes in colors of the points on the initial rectangle shape. When we color the points, a value for the hue in the HSV color space is set to the distance $d_i(t)$, which $d_i(t) = 0$ corresponds to $H = 0°$ and $\max_i d_i(t)$ corresponds to $H = 360°$. We set values of both S and V to 100%.

3. Numerical experiments

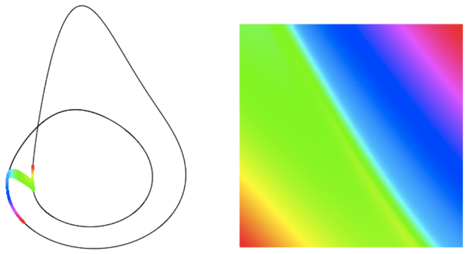
We conduct numerical experiments for well-known systems which are the Rössler system [9]: $\dot{x} = -(y+z),\ \dot{y} = x + 0.2y,\ \dot{z} = 0.2 + z(x - c)$, the chaotic Lorenz system [10]: $\dot{x} = -10x+10y,\ \dot{y} = -xz+28x-y,\ \dot{z} = xy-8z/3$, the chaotic Chua circuit [11, 12]: $\dot{x_1} = 9(x_2 + 5x_1/7 + 3(|x+1| - |x-1|)/14),\ \dot{x_2} = x_1 - x_2 + x_3,\ \dot{x_3} = -10x_2/49$, and the Langford equations [8] to generate a two-torus: $\dot{x} = (z - 0.7)x - 3.5y,\ \dot{y} = 3.5x + (z - 0.7)y,\ \dot{z} = 0.6 + z - z^3/3 - (x^2 + y^2)(1 + z/4)$. In the numerical experiments, we generate the period-2 ($c = 3.5$), period-4 ($c = 4$), period-8 ($c = 4.18$), and chaotic ($c = 5.7$) attractors from the Rössler system for comparison. We integrate the dynamical systems with the fourth-order Runge-Kutta method. The temporal step Δt is set to 0.01 for the Rössler system, the Langford equation, and the Chua circuit, and Δt is set to 0.005 for the Lorenz system.

Figures 1 and 2 show examples of rectangles which are evolved by the dynamical systems. The initial rectangles colored according to distances between a center point and points on the rectangle are also depicted in the right hand side of the figures. From Figs. 1(a), 1(b), 1(c), and 2(a), if the attractors are periodic or a two-torus, no stripe patterns appear. On the other hand, if the dynamics is chaotic, the dense stripe patterns are formed due to the stretch-and-fold mechanism [7] as shown in Fig. 2 (b), (c), and (d). From there results, it is shown that the stretch-and-fold mechanism of the three-dimensional systems can be described on the rectangles, or the two-dimensional image.

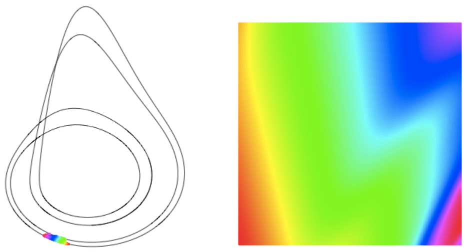
To evaluate the stripe patterns on the initial rectangle quantitatively, we investigate the changes of the number of segments which consist of points whose colors are the same each other and the distribution of the number of the points which are contained in each segment. To calculate them, we firstly divided the initial rectangles into multiple segments according to the colors of the points on the initial rectangles.

Let θ be the threshold which is defined by $|\max_i d_i(t) - \min_j d_j(t)|/N_c$, where the number of colors N_c is a parameter to determine the threshold. If nearby two points $\boldsymbol{r}_i(t_0)$ and $\boldsymbol{r}_j(t_0)$ on the initial rectangle have the same color at time t, or $k\theta + d_{min}(t) \leq d_i(t), d_j(t) < (k+1)\theta + d_{min}(t)$ $(d_{min}(t) \equiv \min_l d_l(t),\ k = 0, 1, \ldots, N_c - 1)$, they are contained in the same segment. Through the numerical experiments, we set the parameter N_c to eight.

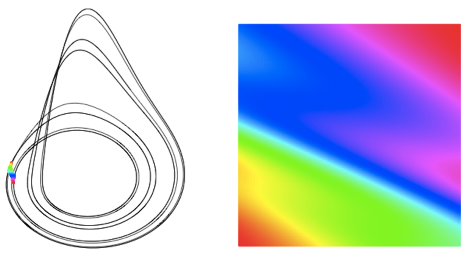
Figure 3 shows how the number of the segments changes with time. From Fig. 3, the number of the segments quickly increases with time if the dynamics is chaotic, but the number of the segments dose not increase in the case of the periodic attractors and the two-torus. These results imply that the speed of increases of the number of the segments might depends on the maximum Lyapunov exponent of the dynamical systems. When the rectangles are stretched and folded repeatedly by the chaotic dynamics, the number of the points contained in each segment becomes small with



(a) Period-2



(b) Period-4

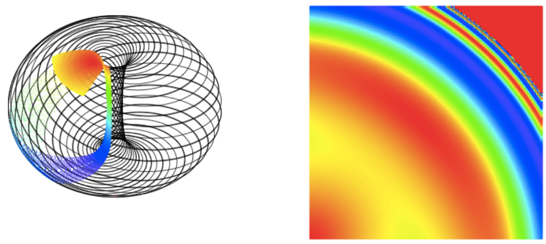


(c) Period-8

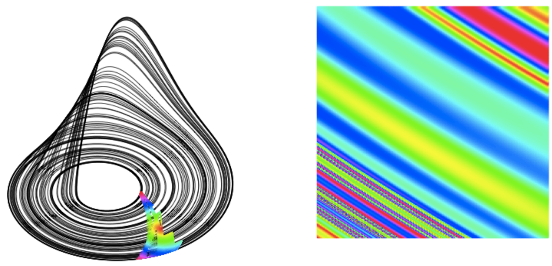
Figure 1: Examples of the rectangles for (a) the period-2, (b) the period-4, (c) the period-8, and the number of points on the rectangle is 40,000.

time and approaches to the number of the points on the rectangle. Then, the stripe patterns on the initial rectangles disappear.

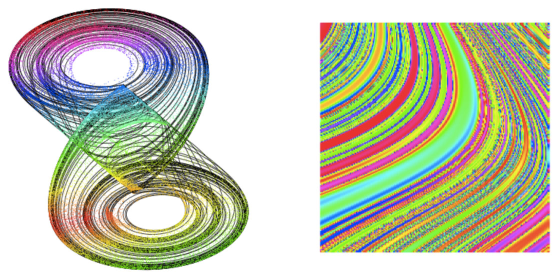
Figure 4 shows the distribution of the number of the points which are contained in the same segments in the case of the chaotic dynamical systems, which is described in the logarithmic scale. From Fig. 4, the distributions obey the power law. The slope of the lines in Fig. 4 is about two, which is a similar value to the fractal dimension of these chaotic attractors. The reason is that the number of times that the rectangles are stretched and folded relates to the number of the segments on the rectangles. Because of this, the distributions of the number of the points in the segments might reflect the fractal dimension of the chaotic attractors.



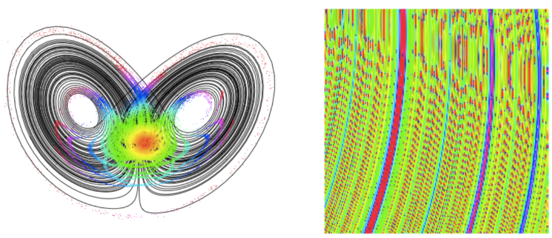
(a) Two-torus



(b) Rössler attractor



(c) Double scroll attractor



(d) Lorenz attractor

Figure 2: Examples of the rectangles for (a) a two-torus from the Langford equation ($T = 3,000$), (b) the chaotic attractors from the Rössler system ($T = 3,000$), (c) the chaotic Lorenz attractor ($T = 2,000$), and (d) the double scroll attractor from the Chua circuit ($T = 3,000$). The number of points on the rectangle is 40,000.

4. Conclusion

In this paper, we proposed a simple visualization method for the stretch-and-fold mechanism in chaotic dynamics. We firstly applied the proposed method to three-dimensional chaotic dynamical systems and showed that the stretch-and-fold mechanism can be visualized by the stripe patterns on the rectangles. One of the advantages of our method is that we can describe the stretch-and-fold mechanism of the chaotic dynamics on the two-dimensional rectangle. In addition, through the analysis of the stripe patterns, we quantitatively evaluated the formation of the stripe patterns and discussed the speed of increases of the number of segments on the rectangle and the distribution of the number of points contained in each segment. As a future work, we will apply our method to higher dimensional chaotic dynamical systems.

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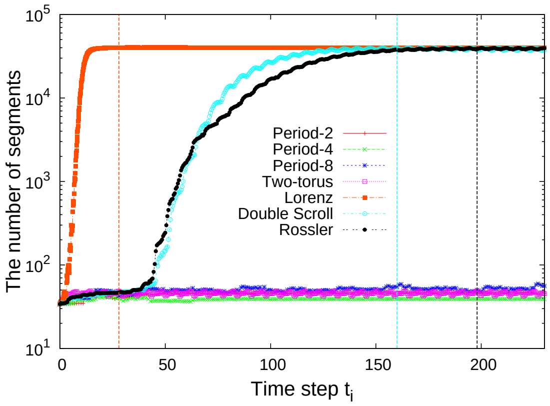

Figure 3: The changes in the number of the segments with time. The number of points on the rectangle is 40,000. The horizontal axis shows the time step $t_i = \Delta t \times i$ $(i = 1, \ldots, 24, 000)$. The vertical lines show the first time when the number of the segments is equal to the number of points on the rectangle for the results of the chaotic attractors.

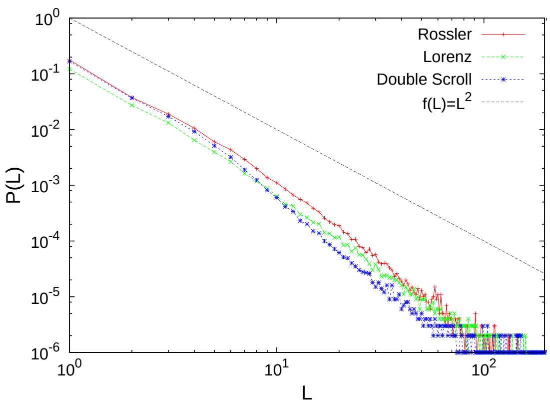

Figure 4: The distribution of the number of points contained in each segment L. The number of points on the rectangle is 129,600. The time T of the Rössler system and the Chua circuit is $3, 000$, and that of the Lorenz system is $1, 500$.

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