IEICE Proceeding Series

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Vol. 2 pp. 439-440
Publication Date: 2014/03/18
Online ISSN: 2188-5079

2013 International Symposium on Nonlinear Theory and its Applications
NOLTA2013, Santa Fe, USA, September 8-11, 2013

Guaranteed high precision estimation for interpolation error constant

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Abstract—For various interpolation error constants that appear in the numerical analysis of the finite element method and so on, we consider to give a uniform framework to estimate these constants with high precision.

1. Introduction

The problem to estimate the constants is in fact to bound the eigenvalues of certain differential operators. Our proposed framework contains two parts: first, based on the finite element method, we give bounds for the leading eigenvalues of the differential operators in consideration; second, high precision bounds are provided by further adopting the Lehmann-Goerisch method. In this paper, we focus on the first step.

2. Error constants in Rayleigh quotient form

There are various interpolation error constants for the interpolation functions(see, e.g., [1, 4]), most of which can be characterized by the Rayleigh quotient form,

$$C^{-2} := \inf_{u\in V} R(u), \quad R(u) := \frac{N(u,u)}{D(u,u)}, \tag{1}$$

where V is certain function space; $N(\cdot,\cdot)$ and $D(\cdot,\cdot)$ are positive symmetric bilinear forms; $D(\cdot,\cdot)$ is positive definite. Define $|\cdot|_D := \sqrt{D(\cdot,\cdot)}$ and $|\cdot|_N := \sqrt{N(\cdot,\cdot)}$. Let's denote the stationary point and values of $R(u)$ by $\lambda_1 \le \lambda_2 \cdots$. A direct fact is that $C^2 = 1/\lambda_1$.

Below we show several constants defined on a triangle T, for which the edges and nodes are denoted by $\{e_i\}$, $\{p_i\}$ $(i = 1, 2, 3)$, respectively.

Example 1: Let us consider the constant appearing in the following interpolation estimation:

$$\|u - \pi_0 u\|_0 \le c_{01}|u|_1 \tag{2}$$

where $\pi_0 u$ is constant function such that

$$\int_{e_1} u - \pi_0 u ds = 0 .$$

In case of K being isosceles right triangle, the optimal constant c in (2) is just the Babuska-Aziz constant. The constant c_{01} is characterized by

$$c_{01}^{-1} := \inf_{v\in H^1(T)} \frac{|u|_1}{\|u - \pi_0 u\|_0}$$

In this case, we can choose the terms in (1) to be : $V = H^1(T) \setminus P^0(T)$,

$$D(u,u) = \|u - \pi_0 u\|_0^2 , \quad N(u,u) = |u|_1^2 .$$

Example 2: Let us consider the interpolation constant for Fujino-Morley interpolation:

$$|u - \pi_2 u|_1 \le c_{11}|u|_2 , \tag{3}$$

where $\pi_2 u$ is a quadratic polynomial such that

$$\int_{e_i} \nabla(u - \pi_2 u)\cdot n ds = 0 \quad (i = 1, 2, 3) ,$$
$$u(p_i) - \pi_2(p_i) = 0 \quad (i = 1, 2, 3) .$$

The constant c_{11} is characterized by

$$c_{11}^{-1} := \inf_{v\in H^2(T)} \frac{|u|_2}{|u - \pi_2 u|_1}$$

Thus, to characterize the constant c in the form (1), we choose $V = H^2(T) \setminus P^1(T)$ and

$$D(u,u) := |u - \pi_2 u|_1^2, \quad N(u,u) := |u|_2^2$$

3. Lower bound of eigenvalues

Let us calculate the eigenvalues λ_i of (1) approximately. Suppose V^h is a finite dimensional space over triangulation T^h of T , which may not be a subspace of V. The discrete eigenvalues characterized by Rayleigh quotient R over V^h are denoted by $\lambda_{1,h} \le \lambda_{2,h} \cdots$.

Let $P_h : V \to V^h$ be a projection such that

$$N(u - P_h u, v_h) = 0, \quad \forall v_h \in V^h$$

and suppose an error estimation for P_h is

$$|u - P_h u|_D \le M_h |u - P_h u|_N$$

Then we have the following theorem to bound the eigenvalues from below.

Theorem 1 *If $\lambda_{k,h} M_h < 1$, $k < \dim(V^h)$, then*

$$\frac{\lambda_{k,h}}{1 + \lambda_{k,h} M_h^2} \le \lambda_k, (i = 1, \cdots, k)$$

Remark 1: The proof is a simple extension of the method developed in our latest paper [2] or [3], where the projection P_h is a generalized one.

Remark 2: Since we do not require $V^h \subset V$, one can not expect that $\lambda_{k,h}$ gives an upper bound for λ_k.

4. Construction of V_h and P_h

To apply Theorem 1, we need to define V^h and P_h for each interpolation.

For the interpolation π_0, we choose V^h to be the piecewise linear conforming FEM space. Denoted by $P_{h,1}$ the corresponding projection,

$$(\nabla(u - P_{h,1}u), \nabla v_h) = 0, \ \forall v_h \in V^h \ ,$$

$$\int_{e_1} P_{h,1}u - u ds = 0.$$

Noticing that $\pi_0(u - P_{h,1}u) = 0$, we consider

$$\|u - P_{h,1}u\|_0 \le M_{h,1}|u - P_{h,1}u|_1 .$$

The estimation for $M_{h,1}$ can be estimated by using the result in [1].

For the interpolation π_2, we choose V_h to be

$$V^h := \Big\{ v_h \mid \text{on each } K \in T^h, v_h|_K \in P_2(K);$$

$$\int_e \frac{\partial v_h}{\partial n} ds \text{ is continuous on inner edges } e;$$

$$v_h \text{ is continuous at the nodes. } \}$$

The Rayleigh quotient over V^h is defined by using new bilinear forms: for $u_h \in V^h$,

$$D_h(u_h, u_h) = \sum_{K \in T^h} \int_K |\nabla(u_h - \pi_2 u_h)|^2 dxdy \ ,$$

$$N_h(u_h, u_h) = \sum_{K \in T^h} \int_K |\nabla^2 u_h|^2 dxdy \ .$$

Notice that $D_h(\cdot,\cdot) = D(\cdot,\cdot)$, $N_h(\cdot,\cdot) = N(\cdot,\cdot)$ for $u \in H^2(T)$.

The corresponding projection $P_{h,2}$ is taken to be the Fujino-Morley interpolation of u on each element K, that is,

$$(P_{h,2}u)|_K = \pi_2(u|_K) \ ,$$

which has the property that for $u \in H^2(T)$

$$N_h(u - P_{h,2}u, v_h) = 0, \quad \forall v_h \in V^h \ .$$

We further need the following estimation

$$|(u - P_{h,2}u) - \pi_2(u - P_{h,2}u)|_1 \le M_{h,2}|u - P_{h,2}u|_2 \ .$$

Since $\pi_2(u - P_{h,2}u) = 0$, the above estimation is

$$|u - P_{h,2}u|_1 \le M_{h,2}|u - P_{h,2}u|_2 \ ,$$

where the quantity $M_{h,2}$ can be bounded by adopting the Babuska-Aziz technique.

5. Sharpen upper bounds by applying Lehmann-Goerisch's theorem

Once we have a rough lower bound ν for λ_{k+1} by using FEM, such that $\lambda_k < \nu \le \lambda_{k+1}$, we can then apply the Lehmann-Goerisch's theorem to polynomial bases to obtain well improved bounds for leading eigenvalues $\lambda_1, \cdots$, λ_k [5]. Thus a much sharped upper bound for the constants become available. The detailed talk on this method will be omitted here.

Summary: In this paper, we proposed a uniform framework to bound the interpolation constants with high precision. The computation result and the application to other interpolations will be given in the subsequent presentations.

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