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Entropy and Asymptotic Behavior of Cellular automataa

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Abstract—In this paper the author discusses asymptic behavior of cellular automata and their spatial thermodynamic quantities, entropy and topological pressure.

1. Introduction

Cellular automata as a class of dynamical systems show various fascionating orbit structure. From the viewpoint of ergodic theory, we can understand cellular automata as factors of subshifts [1]. Furthermore, subshifts have also a backgrounds of statistical physics. Though classification of attractor structure of cellular automata is well described in [2] and [3], asymptotics of thermodynamic quantities, entropy and topological pressure, had not been studied because it is hard to estimate the quantities in infinite-to-one factor case.

In this paper the author estimate asymptotics of topological pressure for a class of cellular automaton map which has bounded interaction radius.

2. Subshifts and thermodynamic formalism

Let $S = \{0, 1, \ldots, s-1\}$ be an alphabet set and $\Sigma = S^{\mathbb{Z}}$. Shift on Σ is defined for each $x = (x_i) \in \Sigma$ by $(\sigma x)_i = x_{i+1}$. We call the pair (Σ, σ) full s shift. Open set family is generateed by cylinder sets $[a_1 \cdots a_k] = \{x \in \Sigma; a_i \in S, x_i = a_i\, i = 1, \cdots, k\}$ and by the topology Σ becomes a compact metrizable space. It is easy to see that σ on Σ is a homeomorphism. For σ-invariant subset $X \subset \Sigma$, restriction of σ on X is also a homeomorphism. We call the pair (X, σ) a subshift.

For a given subshit X, a set $W_n(X)$ denote all admissible words of length n in X and $W_(X) = \cup_n W_n(X)$ word set of X. $Fix_n(X) = Fix_n(X, \sigma)$ denote all fixed points $\sigma^n x = x$.

Typical example of subshift is Markov subshift. Let M be a structure matrix of size s and $X = \{x \in \Sigma; M_{x_i x_{i+1}} = 1\}$. In this case $\#Fix_n(X)$, cardinarity of $Fix_n(X)$, equals to the sum of all elements of M^n.

Sofic subshifts are an extension of Markov subshifts. We call a subshift sofic subshift if and only if its word set is generated by a regular language. Markov subshifts are also sofic subshifts.

Factor map τ of a subshift (X, σ) is a shift commute continuous map. In other words, $\tau(\sigma x) = \sigma(\tau x)$ and continuous. Because factor of sofic subshift is also sofic, sofic subshifts are a class of subshifts on which cellular automata acts.

If a factor map is onto, it is a finite to one map. If a factor map is not onto, it is a uncountably infinite to one map. The second case means that transision dynamics of cellular automata is hard to analyze as dynamical systems.

For a irreducible sofic subshift (X, σ) and continuous function U on X, topological pressure $P(U)$ is defined as follows:

$$P(U) = P(U, X, \sigma) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in Fix_n(X)} e^{-S_n U(x)}$$

where $S_n U(x) = \sum_{k=0}^{n-1} U(\sigma^k x)$. From the context of statistical physics, $P(U)$ corresponds free energy.

Let μ be a shift invariant Borel probability measure on X and α be a finite measurable partition of X. Entropy with respect to α is $H(\alpha) = \sum_{A\in\alpha} -\mu(A) \log \mu(A)$. Entropy of the triplet (X, σ, μ) is defined by

$$h_\mu(X, \sigma) = \sup_\alpha \lim_{n\to\infty} \frac{1}{n} H(\alpha \vee \sigma^{-1}\alpha \vee \cdots \sigma^{-(n-1)}\alpha).$$

Note that the supremum atains if α is a generating partition.

From the measure theoretic entropy we define topological entropy $h(X, \sigma)$ by

$$h(X, \sigma) = \sup_\mu h_\mu(X_\sigma)$$

and equals to $\lim_n \frac{1}{n} \log \#Fix_n(X, \sigma)$. In case of our settings $h(X, \sigma) = P(0) = P(X_n, \sigma)$.

For topologically mixing sofic subshifts with invariant Borel probability measure ν (X, σ, ν) we have the following variational principle.

$$P(U) = \max_{\nu:invariant} h_\nu(X, \sigma) - \int_X U d\nu$$

The equality holds if μ is Gibbs measure with U bounded variation:

$$C^{-1} \leq \frac{\mu([x_0 x_1 \ldots x_{n-1}])}{exp(-nP(U) - S_n U(x))} \leq C$$

where C is a constant.

3. Zeta function and Tower representation

For a sofic subshift (X, σ) the definition of Ruelle's dynamical zeta function with potential U is the following:

$$\zeta(z) = \exp(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in Fix_n(X)} e^{-S_n U(x)}).$$

It is well known that between the convergence radius r of zeta function and topological pressure $P(U)$ we have $P(U) = -\log r$.

Let B be a subset of $W(X)$ and assume B generates $W(X)$ by concatenation of words, we can construct the other dynamical system $(B^{\mathbb{Z}} \times \mathbb{N}, \sigma_T)$ which is equivalent to (X, σ).

For $y = (y_i) \in B^{\mathbb{Z}}$ and $n \le |y_0|$,

$$\sigma_T(y, n) = \begin{cases} (y, n+1) & (n+1 < |y_0|) \\ (\sigma y, 0) & (n+1 = |y_0|) \end{cases}$$

By the representation, Ruelle's dynamical zeta function $\zeta(z)$ of (X, σ) is the reciplocal of

$$D(z) = 1 - \sum_{b \in B} z^{|b|}$$

i.e. $\zeta(z)^{-1} = D(z)$ and if U depends only on the first word of y,

$$D(z) = 1 - \sum_{b \in B} e^{-S_n U(b)} z^{|b|}.$$

Topological pressure $P(U)$ equals convergence radius of zeta function. If (X, σ) is sofic, B is finite because the language of X is regular. So $P(U) = -\log r$ where r is the minimum solution of $D(z) = 0$.

The formula shown in this section is in [4]

4. Entropy formula of factor map

Let $\tau : (X, \sigma) \to (Y, \sigma)$ be a surjective factor map from an ergodic subshift (X, σ, μ) to (Y, σ, ν) where $\nu(\cdot) = \mu(\tau^{-1}(\cdot))$. It is easy to see the entropy formula

$$h_\mu(X, \sigma) - h_\nu(Y, \sigma) = \int_Y h_d d\nu.$$

The function h_d in $L^1(Y, \nu)$ is defined by $\lim_n \frac{1}{2n+1} \log d(y_{-n} \cdots y_n)$ where $d(w) = \#\tau^{-1} w$. The limit exists for almost every $y \in Y$ because subadditive ergodic theorem holds. The equality holds if ν is a Gibbs measure with potential function $U(\tau \cdot)$ where $U(\cdot)$ is a potential function on X.

Though this result is shown in [5], it is hard to estimate the entropy gap for each iteration of cellular automaton map. In the case of spatial topological pressure, we can estimate asymptotics if interaction radius of cellular automaton map is bounded.

5. Asymptotics of topological pressure

If we have Ruelle's dynamical zeta function, we can estimate topological pressure from its convergence radius. In the case of bounded interaction radius, the shift space is well represented by the tower transformation in Section 3. The example below is the typical case of ECA224.

000	001	010	011	100	101	110	111
0	0	0	0	0	1	1	1

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oo....o...o...o..ooo..o.o.oo.ooo.o.oo.oo..
oo................oo...o.o.oo.ooo.o.oo.o..
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Table 1: Rule table of ECA224 and typical orbit. "o" shows alphabet 1 and "." shows alphabet 0.

Example 1 *Let X_0 be full shift on $\{0,1\}$ and τ be the cellular automaton map of ECA224. Set $X_n = \tau^n X_0$ and B_n be the generator of the word set of (X_n, σ) defined in Section 3. Then we have $B_n = (1, 10)^+ 0^+ 0^n \bigcup (1, 10) \bigcup 0$ and*

$$D_n(z) = (1 - z - z^2)(1 - z) - (z + z^2) z^{n+1}.$$

Because the function $D_{B_n}(x)$ is explicitly written we can have asymptotic behavior of topological pressure with constant potential function $P(X_n, \sigma)$ as follows:

$$P(X_n, \sigma) = P(X_0, \sigma) + O(\lambda^n)$$

where $\lambda = (-1 + \sqrt{5})/2$.

To estimate interaction radius of ECA224 is clear and so it is easy to have B_n and ζ_n. The non-trivial case is ECA184.

Example 2 *For ECA184 by observation we have $B_n = (10, 1)^* (10)^n (0, 01)^*$ and*

$$D_n(z) = (1 - z - z^2)^2 - z^{2n} (z + z^2)^2.$$

So we have

$$P(X_n, \sigma) = P(X_0, \sigma) + O(\lambda^n)$$

where $\lambda = (-1 + \sqrt{5})/2$.

The two example show that asymptotics of topological pressure decays exponentially. And their exponent are topological entropy of unstable invariant sets.

000	001	010	011	100	101	110	111
0	0	0	1	1	1	0	1

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Table 2: Rule table of ECA184 and typical orbit. "o" shows alphabet 1 and "." shows alphabet 0. The number of 1 is conservative.

Because ECA184 conserves the number of '1', initial distribution p of "1" plays important role in the asymptotics. To study the effects of the distribution, we see the potential function on the attractor determined by the initial distribution.

If $p = 0$, its initial configuration and attractor are one point $\{0^\infty\}$. Case $p = 1/2$, its attractor is $\{(10)^\infty, (01)^\infty\}$. In the two case we have $P(X_\infty, \sigma) = 0$ because there exists only finite number of points in their attractor. However, $p = 1/2$ case, basin of attraction is also a shift invariant and has entropy $\log 2$.

In case of $0 < p < 1/2$ we have positive topological pressures and $B_\infty = (0^+(10)^+)$. So we have a problem the relationship between p and $P(X_\infty, \sigma)$ with suitable weights. For each $b = 0^m(10)^n \in B_\infty$ we can construct corresponding initial configurations to both term 0^m and $(10)^n$. For term 0^m we only have initial configuration 0^m with probability $(1-p)^m$. For term $(10)^n$ we have initial configuration of "first return path of length $2n$" and the number of them is equals to Catalan number C_n with probability $(p(1-p))^n$. By the obserbarion we have the following result.

Result 1 *Let $B_\infty = (0, 10)$ be the generating word set of attractor $X_\infty = \cup_n \tau^n X_0$ in case of $p < 1/2$. We set a weight function $U(b) = (1-p)^m(p(1-p))^n C_n$ for each $b = 0^m(10)^n \in B_\infty$ where $C_n = \frac{1}{n+1}(2n)!/(n!)^2$ is Catalan number. Then we have*

$$D_\infty(z) = 1 - \frac{qz}{1-qz}\frac{1-\sqrt{1-4pqz^2}}{2pz}$$

and convergence radius is $min(1/q, 1/\sqrt{4pq})$ where $q = 1-p$. Hence, we get the result on topological pressure of (X_∞, σ) with $p < 1/2$

$$P(U) = \begin{cases} -\log(1-p) & (0 \le p < 1/5) \\ \log 2 - \frac{1}{2}(\log p + \log(1-p)) & (1/5 \le p < 1/2). \end{cases}$$

6. Conclusion

In this paper we show three examples of asymptotics of topological pressure. To estimate them we use regular languages of sofic subshift and its zeta function. This framework works well if interaction radius of cellular automaton map is bounded. The problems in the future is to apply this methos to unbounded case.

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