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Estimation of maximum Lyapunov exponent using generalized synchronization in semiconductor lasers with optical feedback

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Abstract—We numerically estimate the maximum Lyapunov exponent for a semiconductor laser with time-delayed optical feedback by using generalized synchronization. We use the auxiliary system approach to observe generalized synchronization, where a chaotic input signal from a semiconductor laser is injected into two semiconductor lasers with optical feedback and the two injected lasers are identically synchronized. The optical injection signal is removed after synchronization, and the two lasers start desynchronized. The maximum Lyapunov exponent can be evaluated by measuring an exponential growth rate of the difference between the outputs of the two desynchronized lasers. The maximum Lyapunov exponent estimated from this method is consistent with that obtained from linear stability analysis. This method can be applied for the estimation of the maximum Lyapunov exponent in experimental systems.

1. Introduction

Semiconductor lasers with optical feedback has infinite dimensionality due to time-delayed feedback and can generate complex chaotic outputs. Optical chaos can be applied to fast physical random number generation [1]. Physical random number generators using semiconductor lasers with optical feedback produce fast random number generation at several gigabits per second since the oscillation frequency of the optical chaos corresponds to a few GHz.

Semiconductor lasers with optical feedback can be treated as nondeterministic systems since the lasers have spontaneous emission. Optical chaos enhances entropy generated from the spontaneous emission, which results in the application to physical random number generators [2, 3]. The entropy rate of optical chaos is considered as a speed of unpredictability. Sampling rates at the speed under the entropy rate guarantee randomness of physical random number generator using optical chaos [2, 3].

It has been numerically shown that the entropy rate almost corresponds to the maximum Lyapunov exponent in a semiconductor laser with optical feedback [2]. Lyapunov exponents are growth rates of small perturbations to an orbit of dynamical systems in the phase space. At least one positive Lyapunov exponent represents deterministic chaos and the Lyapunov exponents determine predictable time of the dynamical systems.

It is difficult to estimate the Lyapunov exponents in time delayed dynamical systems because the systems have infinite dimensionality due to the time delay. On the other hand, a method to determine the predictable time in an optoelectronic feedback system has been recently proposed [4]. The method requires identical synchronization of two systems. However, the method is not applicable for semiconductor lasers with optical feedback because identical synchronization is hardly observed in experiment due to the difficulty of parameter matching.

In this study, we propose a method to evaluate the maximum Lyapunov exponent using generalized synchronization in semiconductor lasers with optical feedback. We use three semiconductor lasers with optical feedback as shown in Fig. 1. The output from one laser (referred to as a drive laser) is unidirectionally injected into the other two lasers (referred to as response 1 and 2 lasers). The system has been known as "auxiliary system [5]." When generalized synchronization is achieved, the two response lasers have identical outputs. A difference between the two outputs exponentially increases after the optical injection is removed. We estimate the maximum Lyapunov exponent from the exponential increase rate of the difference. Moreover, we compare the maximum Lyapunov exponent estimated by using the proposed method with that calculated by using linearized equations of model equations.

2. Numerical model

We use a model consisting of three semiconductor lasers in Fig. 1. The output of drive laser is unidirectionally injected into two response lasers. They have optical feedback and can generate chaos without coupling. The equations of the Lang-Kobayashi model for the drive, response 1 and 2 lasers are described as follows [6, 7, 8],
Drive:

$$\frac{dE_d(t)}{dt} = \frac{1+i\alpha}{2}\left[\frac{G_n(N_d(t)-N_0)}{1+\epsilon|E_d(t)|^2} - \frac{1}{\tau_p}\right]E_d(t) + \kappa_d E_d(t-\tau_d)\exp(-i\omega_d\tau_d) \quad (1)$$

$$\frac{dN_d(t)}{dt} = J_d - \frac{N_d(t)}{\tau_s} - \frac{G_n(N_d(t)-N_0)}{1+\epsilon|E_d(t)|^2}|E_d(t)|^2 \quad (2)$$

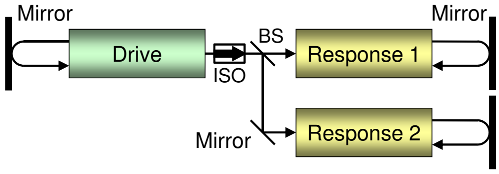

Figure 1: Model of three semiconductor lasers with optical feedback. Identical two lasers (Response 1 and 2 lasers), starting from different initial conditions, are subjected to optical injection from another laser (Drive). BS is a beam splitter. ISO is an optical isolator.

Response 1 and 2:

$$\begin{aligned} \frac{dE_{1,2}(t)}{dt} &= \frac{1+i\alpha}{2}\left[\frac{G_n(N_{1,2}(t)-N_0)}{1+\epsilon|E_{1,2}(t)|^2}-\frac{1}{\tau_p}\right]E_{1,2}(t) \\ &\quad + \kappa_r E_{1,2}(t-\tau_r)\exp(-i\omega_r\tau_r) \\ &\quad + \sigma E_d(t-\tau_{inj})\exp(i\Delta\omega t) \\ &\quad + \xi_{1,2}(t) \end{aligned} \tag{3}$$

$$\begin{aligned} \frac{dN_{1,2}(t)}{dt} &= J_r - \frac{N_{1,2}(t)}{\tau_s} \\ &\quad - \frac{G_n(N_{1,2}(t)-N_0)}{1+\epsilon|E_{1,2}(t)|^2}|E_{1,2}(t)|^2 \end{aligned} \tag{4}$$

E and N is the complex electric field amplitude and the carrier density respectively. Subscripts d, 1, and 2 represent Drive, Response 1 and 2 lasers, and the parameters for the Response lasers have subscript r. G_n, N_0, ϵ, τ_p, τ_s, and α are the fixed parameters and represent the gain coefficient, the carrier density at transparency, the gain saturation coefficient, the photon lifetime, the carrier lifetime, and the linewidth enhancement factor, respectively. κ, J, τ, σ, τ_{inj}, ω, and $\Delta\omega$ represent the feedback strength, the injection current, the feedback delay time, the injection strength, the propagation delay time from the drive laser to the response lasers, the optical angular frequency, and the optical angular frequency detuning between the drive and response lasers, respectively. In our study, $\kappa = 6.2$ ns^{-1}, $J = j \cdot J_{th} = 1.36 J_{th}$, $\tau = 4$ ns are used. Important parameters for generalized synchronization in Fig. 1 are the injection strength σ and the optical frequency detuning $\Delta f = f_d - f_r = \Delta\omega/(2\pi) = (\omega_d - \omega_r)/(2\pi)$, where f is the optical frequency. Generalized synchronization is achieved on negative Δf and large σ, and we set $\Delta f = -4$ GHz and $\sigma = 31.1$ ns^{-1}, which is about five times larger than the feedback strength κ.

The response lasers have spontaneous emission terms $\xi_{1,2}$. The response lasers show different outputs after a short time due to the spontaneous emission term even though the response lasers have identical synchronized states. $\xi_{1,2}$ is the complex number and consists of the real $\zeta_{1,2}$ and the imaginary part $\eta_{1,2}$ as follows,

$$\xi_{1,2}(t) = \zeta_{1,2}(t) + i\eta_{1,2}(t) \tag{5}$$

where $\zeta_{1,2}$ and $\eta_{1,2}$ represent white Gaussian noise described as follows,

$$\langle \zeta_i(t)\zeta_i(t-T)\rangle = D\delta(T), \tag{6}$$

$$\langle \eta_i(t)\eta_i(t-T)\rangle = D\delta(T), \tag{7}$$

where $< \cdot >$ indicates the average over time from minus to plus infinity. δ and D represent the delta function and the noise strength respectively. For simplicity, we restrict our consideration to a single noise strength D that affects the complex electric field. For numerical implementation of the above equations, the BoxMuller algorithm is used to generate white Gaussian noise from two random numbers uniformly distributed on the unit interval. The random numbers are generated with the Mersenne-Twister algorithm.

3. Numerical results

Our method for evaluating the maximum Lyapunov exponent using generalized synchronization utilizes the evolution of the difference between the two response lasers after the optical injection from the drive laser is removed. Figure 2(a) shows the temporal waveform of the drive, response 1 and 2 lasers. The optical injection from the drive laser is injected into the two response lasers until $t = 10$ ns, and generalized synchronization is achieved. The optical injection is removed at $t = 10$, and the two response lasers show different temporal outputs afterwards. The temporal waveform of the absolute difference between the two response lasers is shown in Fig. 2(b). The absolute difference is averaged with the moving window $T_A = kh$ (h is a integration time step) as follows,

$$A_I(t) = \frac{1}{k}\sum_{i=0}^{k-1}|\Delta I(t-ih)| \tag{8}$$

Figure 2(b) shows the temporal waveform of $A_I(t)$. T_A = 5 ns and 0.5 ns are used for the black curve and the gray curve, respectively. For the case of T_A = 5 ns, $A_I(t)$ has a value below 10^{-3} at $t < 10$ ns, where the two response lasers have the optical injection from the drive laser. $A_I(t)$ gradually increases after the optical injection is removed at $t = 10$ ns. $A_I(t)$ exceeds to 1 at $t = 23$ ns and seems to be saturated at $t > 25$ ns. The mean value of $A_I(t)$ after the saturation is about 1.9.

We explain the method to calculate the maximum Lyapunov exponent from Fig. 2(b). Lyapunov exponents denote growth rates of small perturbations to an orbit in the phase space. We can evaluate the maximum Lyapunov exponent from an exponential growth rate of $A_I(t)$. We fit an exponential relation to the curve of $A_I(t)$ in the region where $A_I(t)$ exponentially increases after the optical injection is removed, as follows,

$$A_I(t) \approx C \cdot \exp[\lambda_k(t_1 - t_0)], \tag{9}$$

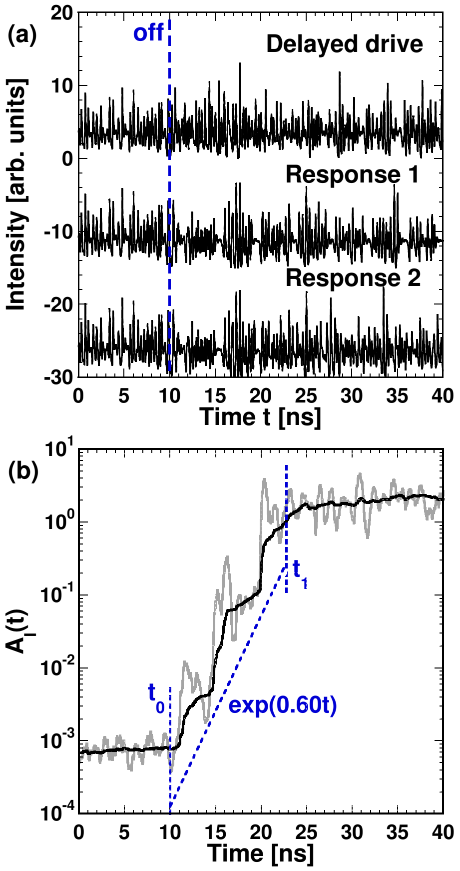

Figure 2: (a): Temporal waveforms of the drive, response 1, and response 2 lasers. The temporal waveform of the drive laser is delayed with the propagation delay time τ_{inj} = 4 ns. The optical injection from the drive is removed at 10 ns. After the removal, the response 1 and 2 start to show different behaviors. (b) The averaged difference $A_I(t)$ between the outputs of the two response lasers shown in (a). The difference is averaged with the intervals T_A. The black and gray curves represent $A_I(t)$ with T_A = 5 ns and 0.5 ns, respectively.

where C is an arbitrary constant. t_0 and t_1 are start and end times to determine the range for exponential fitting of $A_I(t)$. t_0 is the time when the optical injection is removed, and t_1 is the time when $A_I(t)$ reaches 1. λ_k represents an exponential growth rate and λ_k corresponds to the Lyapunov exponent. From the exponential fitting, λ_k = 0.60 is obtained in Fig. 2(b). However, λ_k depends on the locations of the phase space. Therefore, we repeat the calculation of λ_k and obtain the maximum Lyapunov exponent by averaging the exponential growth rate λ_k as follows,

$$\lambda_{max} = \frac{1}{N}\sum_{k=1}^{N}\lambda_k, \qquad (10)$$

where N is the repetition number of the calculation for λ_k. To calculate λ_k repeatedly, it is necessary to repeat the removal and injection of the optical signal from the drive laser. When we achieve generalized synchronization after the optical injection, we need to wait until $A_I(t)$ becomes enough small. In our cases, we use 80 ns to determine the synchronization of the two response lasers after the optical injection.

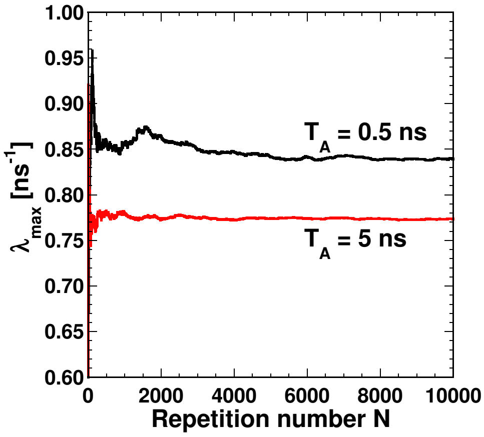

Figure 3: The convergence of λ_{max} as the repetition number N for calculation is changed. T_A = 0.5 ns and 5 ns are used. The convergence values are λ_{max} = 0.84 and 0.77 ns^{-1} for T_A = 0.5 and 5 ns, respectively.

Figure 3 shows the convergence of the maximum Lyapunov exponent λ_{max}. We use T_A = 0.5 ns and 5 ns for the calculation of $A_I(t)$. The maximum Lyapunov exponent converges to λ_{max} = 0.84 ns^{-1} and 0.77 ns^{-1} for T_A = 0.5 ns and 5 ns, respectively. However, the maximum Lyapunov exponent converges to different values for the two T_A. λ_{max} becomes smaller for larger T_A.

4. Dependence of the maximum Lyapunov exponent on feedback strength

We compare the maximum Lyapunov exponent calculated by using our method with that obtained from linearized equations of the Lang-Kobayashi model, to confirm the accuracy of the estimation of the maximum Lyapunov exponent. The method for the calculation of the maximum Lyapunov exponent by using linearized equations in time-delayed dynamical systems is different from the method for ordinary (no time-delayed) dynamical systems [6, 7, 8, 9]. We consider the evolution of the perturbation $\delta\boldsymbol{x}$ to an original chaotic trajectory $\boldsymbol{x}$. We can obtain linearized equations for the perturbation by linearizing the original equations of time-delayed dynamical systems. It is necessary to calculate the norm $d(t)$ of $\delta\boldsymbol{x}$ in the phase space to obtain Lyapunov exponents. All variables within a delay time have been considered as components of a state vector in time-delayed dynamical systems [9, 6]. In numerical simulations, the variables within the delay time are discretized with an integration time step h, and the number of the vari-

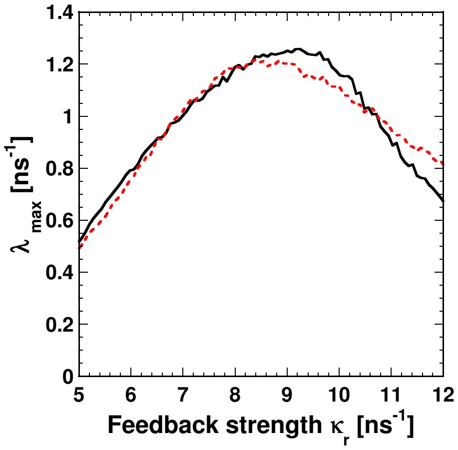

Figure 4: The maximum Lyapunov exponent λ_{max} calculated by using the proposed method (the black solid curve) and the method by using the linearized equations of the Lang-Kobayashi model (the red dashed curve). The feedback strengths κ_r of the two response lasers are changed.

ables within the delay time is $M = \tau/h$, where τ is the delay time. The norm can be calculated by using M variables as follows [6, 7, 8, 9].

$$d(t) = \sqrt{\sum_{i=0}^{M} |\delta \boldsymbol{x}(t - ih)|^2} \qquad (11)$$

The maximum Lyapunov exponent λ_{max} of the dynamical system can be obtained by averaging the logarithm of the change in the norm.

$$\lambda_{max} = \frac{1}{\tau N} \sum_{i=1}^{N} \ln \frac{d(t + i\tau)}{d(t + (i-1)\tau)} \qquad (12)$$

Figure 4 shows the maximum Lyapunov exponents λ_{max} obtained from our method and from the linearized equations as a function of the feedback strength κ_r of the response lasers. The maximum Lyapunov exponent is varied by changing the feedback strength [10]. In Fig. 4, the black solid curve corresponds to the maximum Lyapunov exponent estimated from our method, and the red dashed curve corresponds to the maximum Lyapunov exponent calculated by using the linearized equation of the Lang-Kobayashi model. The black solid curve and the red dashed curve are almost identical in 5 ns$^{-1} \leq \kappa_r \leq 8$ ns^{-1}, which indicates that our proposed method is useful for the estimation of the maximum Lyapunov exponent. Some discrepancies are observed between the two methods at $\kappa_r > 8$ ns^{-1}. For $\kappa_r = 12$ ns^{-1}, the maximum Lyapunov exponent estimated by using our method is $\lambda_{max} = 0.66$ ns^{-1} and has 20% error of the maximum Lyapunov exponent calculated by using linearized equations ($\lambda_{max} = 0.81$ ns^{-1}). However, the black solid curve and the red dashed curve have almost identical for various feedback strengths. Therefore, the proposed method is useful for estimating the maximum Lyapunov exponent. The proposed method can also be applied for the estimation of the maximum Lyapunov exponent in experimental systems.

5. Conclusion

In this study, we proposed a method to estimate the maximum Lyapunov exponent by using generalized synchronization. In the proposed method, three semiconductor lasers with optical feedback are used and the output from one drive laser is unidirectionally injected into two other response lasers, which induce generalized synchronization. The two response lasers with the optical injection have identical outputs. After removing the optical injection, the difference between the outputs of the two response lasers exponentially increases. We can estimate the maximum Lyapunov exponent from the exponential growth rate obtained from exponential fitting of the difference. We estimated the maximum Lyapunov exponent by using the proposed method when the feedback strength is increased and compared the exponent with the maximum Lyapunov exponent calculated by using linearized equations. The maximum Lyapunov exponents calculated by using the two methods agree well with each other. The proposed method is useful for the estimation of the maximum Lyapunov exponent in experimental systems.

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