# Calculation Approach of Effective Permittivity at the Interface of Dielectric Media in FDTD

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### Abstract

*The solutions of effective permittivities at the dielectric interface in the Yee's FDTD algorithm are theoretically investigated by using the boundary conditions. At the interface of two dielectric media, the specific values of these effective permittivities must be determined in order to express the proper solutions from one region into the next with the high accuracy in the FDTD analysis.*

*In electric properties across the discontinuous distribution of two dielectric media, the effective permittivities are chosen to preserve the relations between the field components on adjacent sides of the boundaries. The results of numerical analysis in examples have verified the reasonable agreement between theoretical results and numerical experiments.*

## I. INTRODUCTION

With the rapid development of computer techniques and information technologies in recent decades, Finite- Difference Time-Domain (FDTD) becomes a popular and efficient computational electrodynamics modeling technique. The FDTD technique can allow the user to specify the material at all points within the computational domain [1]. In dealing with complex shape objects, where the boundaries of objects do not coincide with finite-difference grid lines, the staircase approximation is commonly used [2], [3]. However, the staircasing meshing can lead to inaccuracies in electromagnetic simulations. Therefore, a shortcoming can be overcome by using conformal FDTD (CFDTD) techniques [4], [5]. Unfortunately, the above-mentioned CFDTD formulation can only handle cells filled with a homogenous dielectric; hence, it is not well suited for cells containing arbitrarily shaped dielectric objects with two or more dielectric materials. To solve this difficulty, the effective permittivities at dielectric-dielectric interface suitable for second-order accuracy have been obtained by applying of the numerical reflection coefficient [6]. And then, they were theoretically investigated in a rigorous application of the integral form of Maxwell's curl equations on the Yee's lattice leads to the desired values in a straightforward fashion [7].

In this paper, the effective permittivities of the dielectric interface are theoretically proved by using the electric properties of the boundary conditions. Generally, in calculation of effective permittivities, the tangential component of electric field and the normal component of electric flux density are analyzed carefully because of the discontinuity of the fields at the dielectric interfaces.

## II. EFFECTIVE PERMITTIVITIES

In the boundary conditions, it is proved that the tangential component of electric field is continuous across a boundary for which permittivity changes discontinuously. It is also seen that the normal component of the electric flux density is continuous across the discontinuity of electric boundary [8]. Typically, Fig. 1 shows one example of discontinuous distribution of permittivity across interface of two media where the relations between the field components can be obtained as equations (1) and (2) by the boundary conditions. The relations derived above are very useful in determining the discontinuous change of the fields at the boundary.

$$E_{y1}\cos\theta + E_{z1}\sin\theta = E_{y2}\cos\theta + E_{z2}\sin\theta \qquad (1)$$

$$-\varepsilon_1 E_{y1}\sin\theta + \varepsilon_1 E_{z1}\cos\theta = -\varepsilon_2 E_{y2}\sin\theta + \varepsilon_2 E_{z2}\cos\theta \quad (2)$$

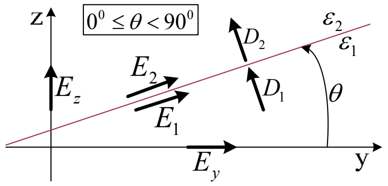


Fig. 1. The electric properties of boundary conditions

In the Fig. 2, schematic drawings of analyzed y-z and x-z planes for effective permittivity calculation are shown in detail. Let consider a boundary surface S separating a region with parameters $\varepsilon_1, E_{y1}, E_{z1}$ one side and $\varepsilon_2, E_{y2}, E_{z2}$ one the adjacent side. Here, $\varepsilon_1$ and $\varepsilon_2$ are the permittivities assigned to the region1 and region2, respectively. $E_{y1}, E_{y2}, E_{z1}$ and $E_{z2}$ are the electric fields with respect to the *y*- and *z*- directions, alternatively. The cell sizes are $\Delta y$

and $\Delta z$. The unknown effective permittivities $\varepsilon_a$ and $\varepsilon_b$ are assigned to $E_y$ and $E_z$ nodes nearest the dielectric interface. The effective permittivity $\varepsilon_a$ associated with the electric component towards $y$- direction $E_y$ at the dielectric interface is firstly considered. At the dielectric interface, in order to obtain expressions for the stored electric flux density $D_y$ in the volume $V_a = \Delta x \cdot \Delta y \cdot \Delta z$, the volume integral of $D_y$ over $V_a$ is taken by using (3).

$$\int_{Va} \varepsilon_a \cdot E_y dV = \int_{Va1} \varepsilon_1 \cdot E_{y1} dV + \int_{Va2} \varepsilon_2 \cdot E_{y2} dV \qquad (3)$$

On the other hand, the electric flux in the volume $V_a$ is obtained by using (4).

$$\int_{Va} E_y dV = \int_{Va1} E_{y1} dV + \int_{Va2} E_{y2} dV \qquad (4)$$

From (3) and (4), the component of $\varepsilon_a$ can be found as follows:

$$\varepsilon_a = \frac{\int_{Va} \varepsilon_a \cdot E_y dV}{\int_{Va} E_y dV} = \frac{\int_{Va1} \varepsilon_1 \cdot E_{y1} dV + \int_{Va2} \varepsilon_2 \cdot E_{y2} dV}{\int_{Va1} E_{y1} dV + \int_{Va2} E_{y2} dV} \qquad (5)$$

In a similar expression of $\varepsilon_a$ calculation, the effective permittivity $\varepsilon_b$ is given by (6).

$$\varepsilon_b = \frac{\int_{Vb} \varepsilon_b \cdot E_z dV}{\int_{Vb} E_z dV} = \frac{\int_{Vb1} \varepsilon_1 \cdot E_{z1} dV + \int_{Vb2} \varepsilon_2 \cdot E_{z2} dV}{\int_{Vb1} E_{z1} dV + \int_{Vb2} E_{z2} dV} \qquad (6)$$

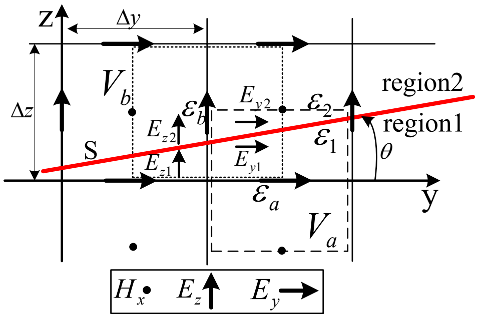
Equations (5) and (6) state that at the interface of dielectric media, the effective permittivities $\varepsilon_a$ and $\varepsilon_b$ are determined in the way to preserve the electric flux and the electric flux density limited by the corresponding volume.

From the expressions of (5) and (6), the relations between the effective permittivities and ratios of the electric fields at the boundary surface S are given by

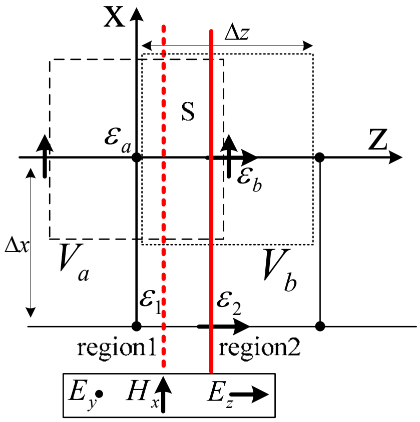
$$\varepsilon_a = \frac{\varepsilon_1 \cdot V_{a1} + \varepsilon_2 \cdot V_{a2}(E_{y2}/E_{y1})}{V_{a1} + V_{a2}(E_{y2}/E_{y1})} \qquad (7)$$

$$\varepsilon_b = \frac{\varepsilon_1 \cdot V_{b1} + \varepsilon_2 \cdot V_{b2}(E_{z2}/E_{z1})}{V_{b1} + V_{b2}(E_{z2}/E_{z1})} \qquad (8)$$

where the ratios of the electric fields $E_{y2}/E_{y1}$ and $E_{z2}/E_{z1}$ can be found by the electric properties of the boundary condition in (1) and (2).



(a) y-z plane



(b) x-z plane

Fig. 2. Schematic drawings of analyzed y-z and x-z planes for the effective permittivity calculation

Since the angle $\theta$ made by the boundary surface S and the x-y plane of FDTD grid is zero, (1) and (2) now become

$$\frac{E_{y2}}{E_{y1}} = 1 \qquad (9)$$

$$\frac{E_{z2}}{E_{z1}} = \frac{\varepsilon_1}{\varepsilon_2} \qquad (10)$$

Substituting (9) and (10) into (7) and (8):

$$\varepsilon_a = \frac{\varepsilon_1 \cdot V_{a1} + \varepsilon_2 \cdot V_{a2}}{V_{a1} + V_{a2}} \tag{11}$$

$$\frac{1}{\varepsilon_b} = \frac{V_{b1}}{V_b}\frac{1}{\varepsilon_1} + \frac{V_{b2}}{V_b}\frac{1}{\varepsilon_2} \tag{12}$$

## III. NUMERICAL EXPERIMENTS AND DISCUSSIONS

We can summarize the above calculated results of the effective permittivities by starting the numerical experiments. It is assumed that an electromagnetic wave is transmitted from dielectric medium 1 ($\varepsilon_1$) to dielectric medium 2 ($\varepsilon_2$) with an incident angle $\theta_i$.

At the interface of two media where a mount of wave is reflected back to the dielectric medium 1 while the other part is transmitted into the dielectric medium 2, the numerical reflection coefficient is calculated by using (11) and (12). In this paper, we assume that the propagation constants of the numerical reflection coefficients are the ideal case. The incident wave is assigned by 374.92 MHz and the cell size is varied to check the accuracy of the effective permittivities.

On the other hand, the calculated numerical reflection coefficient is also compared with calculated results of the conventional method [9] and the theoretical analysis [1].

In [9], the effective permittivities is expressed as

$$\varepsilon_{eff} = \varepsilon_1 \cdot V(i,j,k) + \varepsilon_2 \cdot \left(1 - V(i,j,k)\right) \tag{13}$$

where $V(i,j,k)$ is the fractional volume of the $(i,j,k)$ th cell that contains the dielectric material $\varepsilon_1$, and $\varepsilon_2$ is the dielectric constant of the medium in the remaining volume.

In order to evaluate the calculated accuracy of the effective permittivities while the boundary surface S is orthogonal to the FDTD grid, an example for the effective permittivity calculation is presented by Fig. 3.

By following the proposed method in (11) and (12), we obtain

$$\varepsilon_a = (1/2 + d)\varepsilon_1 + (1/2 - d)\cdot\varepsilon_2 \tag{14}$$

$$\frac{1}{\varepsilon_b} = \frac{\mathrm{d}}{\varepsilon_1} + \frac{1\text{-}\mathrm{d}}{\varepsilon_2} \tag{15}$$

where $d$ is the offset ratio satisfied the condition $0 \le d \le 1/2$. The permittivities $\varepsilon_1$ and $\varepsilon_2$ are assigned to the region1 and the region2, respectively.

By employing the solutions of (14) and (15), the effective permittivities $\varepsilon_a$ and $\varepsilon_b$ are used to compute the numerical reflection coefficients and the calculated results are compared with that of the conventional method and the exact result of the theoretical analysis, as shown in Fig. 4. In here, the numerical reflection coefficient is considered in the change of cell size in x-, y-, z- direction, respectively.

From the numerical experiments in Fig. 4, it is easily shown that the calculated reflection coefficients are good agreement with the theoretical calculation in variation of incident angle. The comparisons show the proposed method as an exact method in the calculation of the effective dielectric constants with high accuracy.

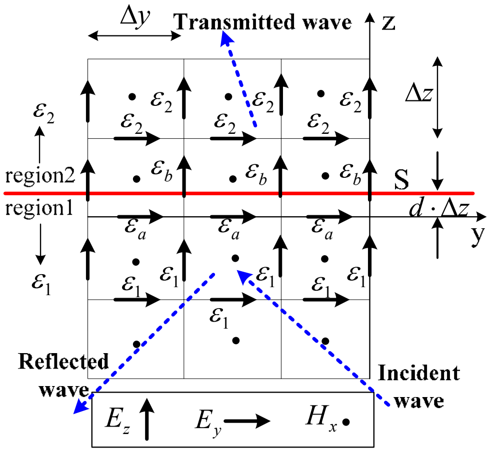
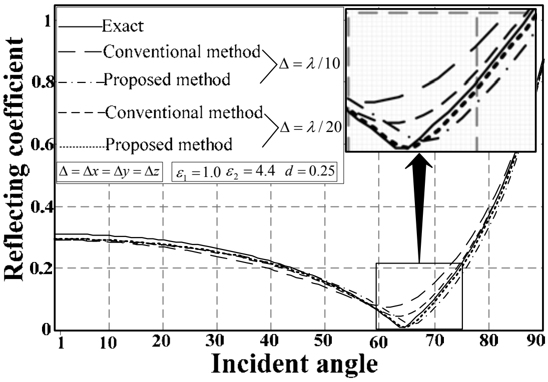
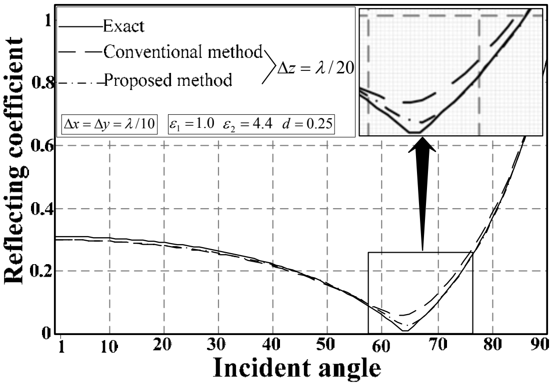


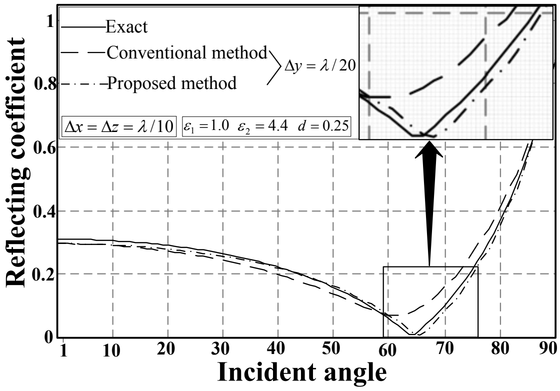
Fig. 3. Example for the effective permittivity calculation



(a) The same variation of the cell sizes in x-, y-, z- directions



(b) The variation of the cell size in z- direction



(c) The variation of the cell size in y- direction

Fig. 4. Illustration for comparison of reflection coefficients

## IV. CONCLUSIONS

In this paper, the solutions of effective permittivities at the dielectric interface in the Yee's FDTD algorithm are theoretically investigated by using the boundary conditions.

On the other hand, the accuracy of analyzed results is expressed by the numerical experiments employing the numerical reflection coefficients. Therefore, the consideration of the maintaining of the electric properties in the computation of the effective permittivities leads us to conclude that at the interface of two dielectric media, the boundary conditions as well as the characteristics of wave impedance and wave reflection coefficient are prevented from unphysical phenomena.

The numerical experiments show that the proposed method for the effective permittivity calculation is an exact solution with high accuracy. The good performances of the numerical experiments above confirm that.

## ACKNOWLEDGEMENT

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